

A SECOND-QUANTIZED KOLMOGOROV-CHENTSOV THEOREM

ABDELMALEK ABDESSELAM

Dedicated to the memory of the founder of the École Polytechnique school of constructive quantum field theory, Roland Sénéor (1938–2016)

ABSTRACT. We prove a very general theorem for the pointwise and pathwise multiplication of random Schwartz distributions. The hypothesis is Wilson's operator product expansion. Due to the urgency of the matter, we will release this article in three successive installments. In the first installment, this one, the complete proof of our theorem is given. It partly relies on the general theory of distribution kernels in combination with probability theory which we call the Schwartz-Grothendieck-Fernique theory. Section 3, to be supplied in the second installment, will give a very elementary and self-contained presentation of this theory in the particular case of temperate distributions. The third installment will add Section 7 dedicated to the special study of conformal field theories in various dimensions.

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1. INTRODUCTION

1.1. The problem of pointwise multiplication for Schwartz distributions. As is well-known, the pointwise multiplication of Schwartz distributions is, in general, impossible [112]. In accordance with the intuition expressed in [109, p. 115], any deterministic theorem to this effect must involve a “compensation principle”. Namely, the regularity of one of the factors must compensate for the other’s lack of regularity. Hörmander’s celebrated theorem [70, Theorem 8.2.10] using wave front sets is a beautiful implementation of this compensation principle, in a direction-wise manner. Another instance of this principle is the multiplication theorem in [17, §2.8.1] which uses Bony’s paraproducts [19]. Let $\mathcal{C}^\alpha(\mathbb{R}^d)$ denote the inhomogeneous Besov space $B_{\infty,\infty}^\alpha(\mathbb{R}^d)$. Then the product of smooth functions continuously extends to $\mathcal{C}^\alpha(\mathbb{R}^d) \times \mathcal{C}^\beta(\mathbb{R}^d)$, provided $\alpha + \beta > 0$ holds. In this article, however, the main issue being addressed is the problem of multiplication of *random* Schwartz distributions which live in the generalized Hölder spaces $\mathcal{C}^\alpha(\mathbb{R}^d)$ with α *negative*. Random distributions are not so nice as to sit in relation to each other in a way that satisfies a compensation principle which would allow one to multiply them via such a deterministic theorem. A particularly unfavorable case is when the two factors are the same, i.e., one is trying to take the square (or higher powers) of a random distribution. The probabilistic setting thus brings extra difficulties, but it also comes with a precious advantage: one does not have to multiply all distributions but only *almost all* of them in the sense of the underlying probability measure. One only has to aim for a pathwise and pointwise multiplication. For a long time, the only known rigorous method for doing this was the Wick product construction (see, e.g., [115], [119, §5.1], [54, §8.5] or [31]). In the last few years, however, there has been tremendous advances on this problem which go far beyond the Wick product method. Notable examples of such advances are the theory of regularity structures [57] and the theory of paracontrolled distributions [56] which both take their inspiration in the theory of rough paths [85]. The main result of this article is a *very general* theorem for pointwise and pathwise multiplication of random distributions which can be seen as a useful complement to these two recent theories. Indeed, our result shows that at the heart of the problem is one of the deepest aspects of quantum field theory (QFT): *Wilson’s operator product expansion* (OPE). The latter was discovered in [134] and first appeared in published form in [20] (see also [103, 76, 77]). For a presentation of the OPE from a physical perspective and also aimed towards a mathematical audience, see [137, Lecture 3].

For concreteness, let us consider the simplest instance of the problem of pointwise and pathwise multiplication of random distributions: the squaring of the fractional massless free

field. Let C denote the continuous bilinear form on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decaying smooth functions defined by

$$C(f, g) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \frac{\widehat{f}(\xi) \widehat{g}(\xi)}{|\xi|^{d-2[\phi]}}$$

where $[\phi] \in (0, \infty)$ is a parameter called the *scaling dimension* of the field ϕ . Note that in this article we will write integrals as above, i.e., with the volume element preceding rather than following the integrand. This is hardly avoidable when integrands take several lines to write, as will be the case in this article. Also, we simply write $d\xi$ instead of $d^d\xi$ as the dimensionality will be clear from the context. Our Fourier transform normalization convention is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} dx e^{-i\xi \cdot x} f(x) .$$

By the Bochner-Minlos Theorem, there is a unique probability measure \mathbb{P} on the space of temperate distributions $\mathcal{S}'(\mathbb{R}^d)$ such that

$$\mathbb{E} e^{i\phi(f)} = \exp \left(-\frac{1}{2} C(f, f) \right)$$

for all test functions $f \in \mathcal{S}(\mathbb{R}^d)$. We use ϕ to denote the random distribution in $\mathcal{S}'(\mathbb{R}^d)$. Here the (canonical) probability space is $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \mathcal{S}'(\mathbb{R}^d)$ equipped with $\mathcal{F} = \text{Borel}(\mathcal{S}'(\mathbb{R}^d))$. Contrary to the treatment in [119, 54], our point of view (which follows [42]) is to see $\mathcal{S}'(\mathbb{R}^d)$ as a *topological space* when endowed with the strong topology. Note that, depending on what is most convenient, we will use $\phi(f)$ or $\langle \phi, f \rangle$ for the duality pairing between a distribution ϕ and a test function f . We will however avoid writing the formal integral

$$\int_{\mathbb{R}^d} dx \phi(x) f(x) .$$

When discussing the theory of kernels, involving several sets of variables, it is necessary to employ notation of this kind which names and shames integration variables with the epithet “dummy”. Instead, we will use:

$$\langle \phi(x), f(x) \rangle_x$$

together with the subscript notation introduced by Schwartz for function spaces. For example, the distribution $\phi(x)$ is said to belong to $\mathcal{S}'_x(\mathbb{R}^d)$ in order to emphasize the name of the variable.

In order to define the pointwise square “ $\phi^2(x)$ ”, the most direct approach is to use a mollifier. Take ρ_{UV} or simply ρ to be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} dx \rho(x) = 1$. Fix $L > 1$ and for, $r \in \mathbb{Z}$, define the rescaled function $\rho_r(x) = L^{-rd} \rho(L^{-r}x)$. The convolution $\phi * \rho_r$ is well defined in a pointwise manner and gives us an opportunity to practice the previous notation. Indeed, by definition,

$$(\phi * \rho_r)(x) = \langle \phi(y), \rho_r(x - y) \rangle_y .$$

The result is a function in $\mathcal{O}_{M,x}(\mathbb{R}^d)$. Here, \mathcal{O}_M denotes the space of temperate smooth functions. It is defined by

$$\mathcal{O}_M(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \forall \alpha \in \mathbb{N}^d, \exists k \in \mathbb{N}, \exists K > 0, \forall x \in \mathbb{R}^d, |\partial^\alpha f(x)| \leq K \langle x \rangle^k\} .$$

For our notations we use the Bourbaki convention $\mathbb{N} = \{0, 1, 2, \dots\}$. We denote the Euclidean norm of a vector $x \in \mathbb{R}^d$ by $|x|$, and we write $\langle x \rangle = \sqrt{1 + |x|^2}$ for its “inhomogeneous norm”. For U an open subset of \mathbb{R}^d , we of course use $C^\infty(U)$ to denote the space of (real-valued) smooth functions on U . We remark in passing that $(\phi * \rho_r)(x)$ in fact belongs to the smaller space \mathcal{O}_C [71, Proposition 7, p. 420] but we will not use this here. Since $(\phi * \rho_r)(x)$ converges to $\phi(x)$ in $\mathcal{S}'_x(\mathbb{R}^d)$ when $r \rightarrow -\infty$, it is natural to try to define the square of ϕ as the distribution ϕ^2 whose action on a test function f would be given by

$$\phi^2(f) = \lim_{r \rightarrow -\infty} \int_{\mathbb{R}^d} dx [(\phi * \rho_r)(x)]^2 f(x) .$$

While the integral makes perfect sense, unfortunately, the limit usually does not. Nevertheless, one can define the smeared square $\phi^2(f)$ for *any* $[\phi] > 0$ as a Hida distribution [14] (see [115] for a related result). The outcome, however, is in general not a true random variable or function on $\Omega = \mathcal{S}'(\mathbb{R}^d)$ but a “second-quantized Schwartz distribution” [61, Ch. 8] (see also [62] and [82]). It is to $L^2(\Omega, \mathcal{F}, \mathbb{P})$ what a Schwartz distribution is to $L^2(\mathbb{R}^d)$. By a “second-quantized” Kolmogorov-Chentsov regularity result, we mean showing that what *a priori* is such a generalized functional of the field ϕ is in fact a true random variable. In the simple example under consideration, it has been known for a long time (see, e.g., [115]) that if the covariance kernel is locally integrable, i.e., if $[\phi] \in (0, \frac{d}{2})$, then such a second-quantized regularity holds. Indeed, if one recenters the distribution-valued random variable $[(\phi * \rho_r)(x)]^2 \in \mathcal{O}_{M,x}(\mathbb{R}^d) \subset \mathcal{S}'_x(\mathbb{R}^d)$, then the desired limit exists. Namely, the correct (Wick) square ϕ^2 or rather $:\phi^2:$ is given by

$$:\phi^2:(f) = \lim_{r \rightarrow -\infty} \int_{\mathbb{R}^d} dx [(\phi * \rho_r)(x)]^2 - \mathbb{E} [(\phi * \rho_r)(x)]^2 f(x) \quad (1)$$

with convergence in every $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 1$, and almost surely. This is the simplest case of the Wick product construction.

Let us revisit this simple example in a way that gives the OPE flavor of our general theorem. When $[\phi] \in (0, \frac{d}{2})$, then one has a *pointwise representation* for the covariance C , i.e.,

$$C(f, g) = \int_{\mathbb{R}^{2d}} dx dy \langle \phi(x) \phi(y) \rangle f(x) g(y)$$

where the pointwise correlation $\langle \phi(x) \phi(y) \rangle$ is defined outside the diagonal by

$$\langle \phi(x) \phi(y) \rangle = \frac{\varkappa}{|x - y|^{2[\phi]}}$$

with

$$\varkappa = \pi^{\frac{d}{2}} \times 2^{2[\phi]} \times \frac{\Gamma([\phi])}{\Gamma(\frac{d}{2} - [\phi])}$$

as shown, e.g., in [53, p. 193]. If now one considers a higher order moment, say a fourth order one,

$$\mathbb{E} [\phi(f_1) \phi(f_2) \phi(f_3) \phi(f_4)] = C(f_1, f_2) C(f_3, f_4) + C(f_1, f_3) C(f_2, f_4) + C(f_1, f_4) C(f_2, f_3)$$

for $f_1, \dots, f_4 \in \mathcal{S}(\mathbb{R}^d)$, then one also has a pointwise representation

$$\mathbb{E} [\phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4)] = \int_{\mathbb{R}^{4d}} dx_1 dx_2 dx_3 dx_4 \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4)$$

featuring the pointwise correlation

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{\varkappa^2}{|x_1 - x_2|^{2[\phi]}|x_3 - x_4|^{2[\phi]}} + \frac{\varkappa^2}{|x_1 - x_3|^{2[\phi]}|x_2 - x_4|^{2[\phi]}} + \frac{\varkappa^2}{|x_1 - x_4|^{2[\phi]}|x_2 - x_3|^{2[\phi]}} .$$

Note that, in this discussion, our pointwise correlations are seen as *ordinary functions* on the open subset of \mathbb{R}^{nd} where the points $x_1, \dots, x_n \in \mathbb{R}^d$ are *distinct*. The integrals above are also on this open subset. Define (again at non-coinciding points) the new function

$$\langle : \phi^2 : (x_1) \phi(x_2)\phi(x_3) \rangle = \frac{2\varkappa^2}{|x_1 - x_2|^{2[\phi]}|x_1 - x_3|^{2[\phi]}} .$$

Then one has the asymptotic behavior

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \langle \phi(x_1)\phi(x_2) \rangle \langle \phi(x_3)\phi(x_4) \rangle + \langle : \phi^2 : (x_2) \phi(x_3)\phi(x_4) \rangle + o(1)$$

when $x_1 \rightarrow x_2$ while the three points x_2, x_3 , and x_4 are *fixed*. This is the simplest instance of Wilson's OPE which here would say that, “inside correlations”, one has

$$\phi(x_1)\phi(x_2) = \mathcal{C}_{\phi\phi}^{\mathbb{1}}(x_1, x_2) \times 1 + \mathcal{C}_{\phi\phi}^{\phi^2}(x_1, x_2) : \phi^2 : (x_2) + o(1)$$

with

$$\mathcal{C}_{\phi\phi}^{\mathbb{1}}(x_1, x_2) = \frac{\varkappa}{|x_1 - x_2|^{2[\phi]}}$$

and

$$\mathcal{C}_{\phi\phi}^{\phi^2}(x_1, x_2) = 1 .$$

Our theorem shows how such an OPE, *with precise bounds on the remainder*, allows one to establish convergence in L^p and almost surely for suitably renormalized products as in (1). We do this in a vast setting which can handle Gaussian and non-Gaussian measures, massive and massless fields, anomalous scaling dimensions, logarithmic corrections, finite degeneracy in the dimension spectrum, as well as lack of translation invariance. Much notation and machinery needs to be introduced before stating our theorem precisely in §1.10. This is provided in the following sections.

1.2. Abstract systems of pointwise correlations. Let \mathcal{A} be a finite set which we will call an alphabet and will serve to label fields.

Define the big diagonal

$$\text{Diag}_n = \{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n \mid \exists i \neq j, x_i = x_j\}$$

and the configuration space

$$\text{Conf}_n = (\mathbb{R}^d)^n \setminus \text{Diag}_n .$$

An *abstract system of pointwise correlations* consists in specifying for all $n \geq 0$ and $A_1, \dots, A_n \in \mathcal{A}$ an element of $C^\infty(\text{Conf}_n)$ denoted by

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle .$$

This is a purely symbolic notation. No constituent of the formula has meaning by itself and the whole simply is some smooth function of the tuple (x_1, \dots, x_n) .

We impose that the degenerate $n = 0$ case is taken care of by setting $\langle \emptyset \rangle = 1$.

We assume symmetry, namely,

$$\langle \mathcal{O}_{A_{\sigma(1)}}(x_{\sigma(1)}) \cdots \mathcal{O}_{A_{\sigma(n)}}(x_{\sigma(n)}) \rangle = \langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle \quad (2)$$

for all permutations $\sigma \in \mathfrak{S}_n$.

We assume \mathcal{A} contains a distinguished element $\mathbb{1}$ having the following “forgetful” property. For all $n \geq 0$, and A_1, \dots, A_n in \mathcal{A} we have

$$\langle \mathcal{O}_{\mathbb{1}}(z) \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle = \langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle \quad (3)$$

for all $(z, x_1, \dots, x_n) \in \text{Conf}_{n+1}$.

1.3. Multilinear multilocal enhancement. For $n \geq 0$, let \mathcal{V}_n be the free module over the algebra $C^\infty(\text{Conf}_n)$ with basis \mathcal{A}^n . A basis element indexed by $(A_1, \dots, A_n) \in \mathcal{A}^n$ will be symbolically denoted by $\mathcal{O}_{A_1} \otimes \cdots \otimes \mathcal{O}_{A_n}$. Thus an element $P \in \mathcal{V}_n$ has a unique expression as

$$P = \sum_{(A_1, \dots, A_n) \in \mathcal{A}^n} f_{A_1, \dots, A_n} \mathcal{O}_{A_1} \otimes \cdots \otimes \mathcal{O}_{A_n}$$

where the f_{A_1, \dots, A_n} are in $C^\infty(\text{Conf}_n)$. To any $P \in \mathcal{V}_n$ we associate a function $\langle P \rangle \in C^\infty(\text{Conf}_n)$. Namely, it is the function

$$(x_1, \dots, x_n) \mapsto \langle P(x_1, \dots, x_n) \rangle$$

where, by definition,

$$\langle P(x_1, \dots, x_n) \rangle = \sum_{(A_1, \dots, A_n) \in \mathcal{A}^n} f_{A_1, \dots, A_n}(x_1, \dots, x_n) \langle \mathcal{O}_{A_1}(x_1) \cdots \otimes \mathcal{O}_{A_n}(x_n) \rangle.$$

The latter is not necessarily symmetric in the arguments x_i , $1 \leq i \leq n$. Note that one can give meaning to $P(x_1, \dots, x_n)$ as a C^∞ function $\text{Conf}_n \rightarrow \mathbb{R}^{\mathcal{A}^n}$ but we will not use this point of view.

For $P \in \mathcal{V}_m$ and $Q \in \mathcal{V}_n$ we define their concatenation $P \otimes Q \in \mathcal{V}_{m+n}$ by

$$P \otimes Q = \sum_{(A_1, \dots, A_{m+n}) \in \mathcal{A}^{m+n}} (f_{A_1, \dots, A_m} \otimes g_{A_{m+1}, \dots, A_{m+n}}) \mathcal{O}_{A_1} \otimes \cdots \otimes \mathcal{O}_{A_{m+n}}$$

if

$$P = \sum_{(A_1, \dots, A_m) \in \mathcal{A}^m} f_{A_1, \dots, A_m} \mathcal{O}_{A_1} \otimes \cdots \otimes \mathcal{O}_{A_m}$$

and

$$Q = \sum_{(B_1, \dots, B_n) \in \mathcal{A}^n} g_{B_1, \dots, B_n} \mathcal{O}_{B_1} \otimes \cdots \otimes \mathcal{O}_{B_n}$$

where we used the notation $(f \otimes g)(x_1, \dots, x_{m+n}) = f(x_1, \dots, x_m)g(x_{m+1}, \dots, x_{m+n})$ for $f \in C^\infty(\text{Conf}_m)$ and $g \in C^\infty(\text{Conf}_n)$. We will use the notation

$$\langle P(x_1, \dots, x_m) Q(x_{m+1}, \dots, x_{m+n}) \rangle = \langle (P \otimes Q)(x_1, \dots, x_{m+n}) \rangle \quad (4)$$

for the evaluation of $\langle P \otimes Q \rangle \in C^\infty(\text{Conf}_{m+n})$ on the argument (x_1, \dots, x_{m+n}) , and similarly for higher products $\langle P_1 \otimes \cdots \otimes P_N \rangle$. The latter are unambiguously defined since concatenation is associative. However, concatenation is not commutative: in general $P \otimes Q \neq Q \otimes P$ and

also the smooth functions of the argument (x_1, \dots, x_{m+n}) given by $\langle P \otimes Q \rangle$ and $\langle Q \otimes P \rangle$ are different. Because of the symmetry (2), one has instead $\langle P \otimes Q \rangle = \langle Q \otimes P \rangle \circ \tau$ where τ is the “braiding map” given by

$$\tau(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n}) = (x_{m+1}, \dots, x_{m+n}, x_1, \dots, x_m) .$$

Nevertheless, if for instance P, Q, R belong to \mathcal{V}_2 then

$$\langle P(x_1, y_1) Q(x_2, y_2) R(x_3, y_3) \rangle = \langle R(x_3, y_3) Q(x_2, y_2) P(x_1, y_1) \rangle$$

and similarly for other permutations in \mathfrak{S}_3 . This is the point of the notation (4): braiding maps take care of themselves. As a result, if P_1, \dots, P_N are say in \mathcal{V}_2 ,

$$\left\langle \prod_{i=1}^N P_i(x_i, y_i) \right\rangle$$

is an unambiguously defined function of the x and y arguments. There is no need to indicate an order in which the product must be written. Finally, note that there is a clear notion of subsystem of pointwise correlations corresponding to a subset \mathcal{A}' of the alphabet \mathcal{A} and obtained by only keeping data concerning labels A in \mathcal{A}' .

1.4. OPE structure. An OPE structure consists in an abstract system of pointwise correlations together with some extra data. To $A \in \mathcal{A}$ we associate a number $[A] \in \mathbb{R}$ called the scaling dimension of the field labelled by A . We impose $[\mathbb{1}] = 0$. For $\Delta \in \mathbb{R}$, we let $\mathcal{A}(\Delta) = \{A \in \mathcal{A} \mid [A] \leq \Delta\}$. We assume that for each triple $(A, B, C) \in \mathcal{A}^3$ we have an element \mathcal{C}_{AB}^C in $C^\infty(\text{Conf}_2)$ thus giving rise to a smooth function $\mathcal{C}_{AB}^C(x, y)$ of $(x, y) \in \text{Conf}_2$.

An OPE structure is what is needed in order to formulate physicists’s statements that if one fixes a number Δ then the “operators” \mathcal{O}_A , $A \in \mathcal{A}$, satisfy an operator product expansion

$$\mathcal{O}_A(x) \mathcal{O}_B(y) = \sum_{C \in \mathcal{A}(\Delta)} \mathcal{C}_{AB}^C(x, y) \mathcal{O}_C(y) + o(|x - y|^{\Delta - [A] - [B]}) \quad (5)$$

“inside correlations”.

A precise statement corresponding to such intuition is that, for all $n \geq 0$ and D_1, \dots, D_n in \mathcal{A} and all fixed $(y, z_1, \dots, z_n) \in \text{Conf}_{n+1}$, we have

$$\begin{aligned} & \langle \mathcal{O}_A(x) \mathcal{O}_B(y) \mathcal{O}_{D_1}(z_1) \cdots \mathcal{O}_{D_n}(z_n) \rangle = \\ & \sum_{C \in \mathcal{A}(\Delta)} \mathcal{C}_{AB}^C(x, y) \langle \mathcal{O}_C(y) \mathcal{O}_{D_1}(z_1) \cdots \mathcal{O}_{D_n}(z_n) \rangle + o(|x - y|^{\Delta - [A] - [B]}) \end{aligned}$$

when taking the limit $x \rightarrow y$. The very easy example from §1.1 corresponds to $\mathcal{A} = \{\mathbb{1}, \phi, \phi^2\}$ and $\Delta = [\phi^2] = 2[\phi]$.

The nontriviality of the mathematical statement embodied in Wilson’s OPE resides in the independence of the shape of this asymptotic expansion from the number n and labels D_1, \dots, D_n as well as positions z_1, \dots, z_n for the “spectator fields” $\mathcal{O}_{D_1}(z_1), \dots, \mathcal{O}_{D_n}(z_n)$.

The multilinear multilocal enhancement §1.3 allows a more elegant rephrasing as

$$\langle \text{OPE}(x, y) \mathcal{O}_{D_1}(z_1) \cdots \mathcal{O}_{D_n}(z_n) \rangle = o(|x - y|^{\Delta - [A] - [B]})$$

where $\text{OPE} \in \mathcal{V}_2$ is defined by

$$\text{OPE} = \mathcal{O}_A \otimes \mathcal{O}_B - \sum_{C \in \mathcal{A}(\Delta)} \mathcal{C}_{AB}^C \mathcal{O}_{\mathbb{1}} \otimes \mathcal{O}_C . \quad (6)$$

Note that a subset $\mathcal{A}' \subset \mathcal{A}$ also defines in an obvious manner a sub-OPE structure.

1.5. Probabilistic incarnations. Suppose that our system of pointwise correlations is such that all smooth functions

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle$$

are locally integrable on Diag_n and have suitable moderate growth at infinity so the integrals

$$\int_{\text{Conf}_n} \prod_{i=1}^n dx_i f(x_1, \dots, x_n) \langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle$$

converge absolutely for all test functions $f \in \mathcal{S}(\mathbb{R}^{nd})$. Then it makes sense to talk about what we call a probabilistic incarnation. It is given by a (not necessarily complete) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with $\mathcal{S}'(\mathbb{R}^d)$ -valued random variables \mathcal{O}_A , $A \in \mathcal{A}$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$, the real-valued random variable $\mathcal{O}_A(f)$ has moments of all orders and for all $n \geq 0$, $A_1, \dots, A_n \in \mathcal{A}$ and all $f_1, \dots, f_n \in \mathcal{S}(\mathbb{R}^d)$ one has

$$\mathbb{E}[\mathcal{O}_{A_1}(f_1) \cdots \mathcal{O}_{A_n}(f_n)] = \int_{\text{Conf}_n} \prod_{i=1}^n dx_i f_i(x_1) \cdots f_n(x_n) \langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_n}(x_n) \rangle .$$

We used the same notation \mathcal{O}_A on both sides of the equation but there is no risk of confusion. If \mathcal{O}_A appears in a formula with the expectation symbol \mathbb{E} then we mean an honest random variable from the probabilistic incarnation. If not, then we mean a constituent symbol participating in the definition of a pointwise correlation. Also, when we say \mathcal{O}_A is an $\mathcal{S}'(\mathbb{R}^d)$ -valued random variable, we mean that the map $\Omega \rightarrow \mathcal{S}'(\mathbb{R}^d)$ given by $\omega \mapsto \mathcal{O}_A(\omega)$ is $(\mathcal{F}, \text{Borel}(\mathcal{S}'(\mathbb{R}^d)))$ -measurable. As said earlier, the space of temperate distributions $\mathcal{S}'(\mathbb{R}^d)$ is here equipped with the strong topology, although the weak-* topology would give rise to the same Borel σ -algebra.

1.6. Soft hypotheses: kernel semi-regularity. Regarding the functions \mathcal{C}_{AB}^C we make the following rather mild hypotheses.

We assume that for all $z \in \mathbb{R}^d$ the function $y \mapsto \mathcal{C}_{AB}^C(y, z)$ is locally integrable at $y = z$ and has at most polynomial growth at infinity. Namely, we assume that $\forall z \in \mathbb{R}^d, \exists k \in \mathbb{N}, \exists K > 0, \exists R > 0$ such that for all $y \in \mathbb{R}^d \setminus \{z\}$,

$$|y| > R \implies |\mathcal{C}_{AB}^C(y, z)| \leq K \langle y \rangle^k . \quad (7)$$

As a result, and following Schwartz's subscript notation, $\mathcal{C}_{AB}^C(y, z)$ can be seen as a z -dependent element of $\mathcal{S}'_y(\mathbb{R}^d)$ defined by

$$\langle \mathcal{C}_{AB}^C(y, z), f(y) \rangle_y = \int_{\mathbb{R}^d \setminus \{z\}} dy \mathcal{C}_{AB}^C(y, z) f(y)$$

for all test function $f(y)$ in $\mathcal{S}_y(\mathbb{R}^d)$.

We now make two more assumptions or soft hypotheses.

SH1: For all $g(z)$ in $\mathcal{S}_z(\mathbb{R}^d)$, and all $f(y)$ in $\mathcal{S}_y(\mathbb{R}^d)$ the function

$$g(z) \langle \mathcal{C}_{AB}^C(y, z), f(y) \rangle_y$$

belongs to $\mathcal{S}_z(\mathbb{R}^d)$. Moreover, the resulting map

$$f(y) \mapsto g(z) \langle \mathcal{C}_{AB}^C(y, z), f(y) \rangle_y$$

is a continuous map $\mathcal{S}_y(\mathbb{R}^d) \rightarrow \mathcal{S}_z(\mathbb{R}^d)$.

SH2: For all $g(x, z)$ in $\mathcal{S}_{x,z}(\mathbb{R}^{2d})$, and all $f(y, w)$ in $\mathcal{S}_{y,w}(\mathbb{R}^{2d})$ the function

$$g(x, z) \langle \mathcal{C}_{AB}^C(y, z), f(y, w) \rangle_y$$

belongs to $\mathcal{S}_{x,z,w}(\mathbb{R}^{3d})$. Moreover, the resulting map

$$f(y, w) \mapsto g(x, z) \langle \mathcal{C}_{AB}^C(y, z), f(y, w) \rangle_y$$

is a continuous map $\mathcal{S}_{y,w}(\mathbb{R}^{2d}) \rightarrow \mathcal{S}_{x,z,w}(\mathbb{R}^{3d})$.

Essentially, we are assuming that the kernel $\mathcal{C}_{AB}^C(y, z)$ is semi-regular and of moderate growth in z . This is a pedestrian way of saying

$$\mathcal{C}_{AB}^C \in \mathcal{S}'_y(\mathbb{R}^d) \hat{\otimes} \mathcal{O}_{M,z}(\mathbb{R}^d)$$

in the sense of [113, §4]. Note that there is no need [123, Theorem 50.1 (f)] to specify if the tensor product is $\hat{\otimes}_\pi$ or $\hat{\otimes}_\varepsilon$, since all spaces involved are nuclear [55, Ch. 2, Theorem 10, p. 55]. In fact (SH1) implies (SH2), as a consequence of the Schwartz theory for Volterra composition [113, §4]. An elementary proof will be given later in §3. It is easy to check (SH1) and (SH2) by hand in the case of the fractional massless free field from §1.1. In that case, $\mathcal{C}_{AB}^C(y, z)$ is a constant multiple of the convolution kernel $|y - z|^{[C]-[A]-[B]}$. The functional analytic difficulties in the more general framework of this section are due to *non-translation invariance* (see, e.g., [34, 91, 139] where similar issues arise).

1.7. Hard hypotheses: factorized nearest neighbor bounds. From now on we assume that $[A] \in [0, \frac{d}{2}]$, for all $A \in \mathcal{A}$. We sharpen the bound (7) on the OPE “structure constants” \mathcal{C}_{AB}^C by requiring $\forall A, B, C \in \mathcal{A}, \forall \epsilon > 0, \exists k \in \mathbb{N}, \exists K > 0, \forall (x, y) \in \text{Conf}_2$,

$$|\mathcal{C}_{AB}^C(x, y)| \leq \frac{K}{|x - y|^{[A]+[B]-[C]+\epsilon}} \langle x \rangle^k \langle y \rangle^k. \quad (8)$$

An element in \mathcal{V}_2 is called *OPE-like* if it is of the form (6) for some $A, B \in \mathcal{A}$ and $\Delta \in \mathbb{R}$.

An element in \mathcal{V}_2 is called *CZ-like* if it is of the form

$$\mathcal{O}_A \otimes \mathcal{O}_1 - \mathcal{O}_1 \otimes \mathcal{O}_A$$

for some $A \in \mathcal{A}$. “CZ” stands for Calderón-Zygmund.

For $m, n, p \geq 0$, and for OPE-like elements $\text{OPE}_1, \dots, \text{OPE}_m$, given by

$$\text{OPE}_i = \mathcal{O}_{A_i} \otimes \mathcal{O}_{B_i} - \sum_{C_i \in \mathcal{A}(\Delta_i)} \mathcal{C}_{A_i B_i}^{C_i} \mathcal{O}_1 \otimes \mathcal{O}_{C_i}$$

CZ-like elements $\text{CZ}_{m+1}, \dots, \text{CZ}_{m+n}$ given by

$$\text{CZ}_i = \mathcal{O}_{B_i} \otimes \mathcal{O}_1 - \mathcal{O}_1 \otimes \mathcal{O}_{B_i}$$

and $B_{m+n+1}, \dots, B_{m+n+p} \in \mathcal{A}$, we require

$$\exists \eta > 0, \exists \gamma > 0, \forall \epsilon > 0, \exists k \in \mathbb{N}, \exists K > 0,$$

$$\prod_{i=1}^{m+n} \mathbb{1} \left\{ |y_i - x_i| \leq \eta \min_{j \neq i} |x_i - x_j| \right\} \times \left| \left\langle \prod_{i=1}^m \text{OPE}_i(y_i, x_i) \prod_{i=m+1}^{m+n} \text{CZ}_i(y_i, x_i) \prod_{i=m+n+1}^{m+n+p} \mathcal{O}_{B_i}(x_i) \right\rangle \right| \leq$$

$$K \prod_{i=1}^{m+n+p} \langle x_i \rangle^k \times \prod_{i=1}^{m+n} \langle y_i \rangle^k \times \prod_{i=1}^m \left\{ |y_i - x_i|^{\Delta_i + \gamma - [A_i] - [B_i]} \times \left(\min_{j \neq i} |x_i - x_j| \right)^{-\Delta_i - \gamma - \epsilon} \right\}$$

$$\times \prod_{i=m+1}^{m+n} \left\{ |y_i - x_i|^\gamma \times \left(\min_{j \neq i} |x_i - x_j| \right)^{-[B_i] - \gamma - \epsilon} \right\} \times \prod_{i=m+n+1}^{m+n+p} \left(\min_{j \neq i} |x_i - x_j| \right)^{-[B_i] - \epsilon} \quad (9)$$

for all collections of $2m + 2n + p$ distinct points in \mathbb{R}^d .

We call this the *enhanced factorized nearest neighbor bound* (EFNNB). It includes, as the $m = n = 0$ special case, the *basic factorized nearest neighbor bound* (BFNNB) for pointwise correlations: $\forall \epsilon > 0, \exists k \in \mathbb{N}, \exists K > 0$,

$$|\langle \mathcal{O}_{B_1}(x_1) \cdots \mathcal{O}_{B_p}(x_p) \rangle| \leq K \prod_{i=1}^p \langle x_i \rangle^k \times \prod_{i=1}^p \left(\min_{j \neq i} |x_i - x_j| \right)^{-[B_i] - \epsilon} \quad (10)$$

for all $(x_1, \dots, x_p) \in \text{Conf}_p$. The ϵ is only needed in order to account for eventual logarithmic corrections. In a conformal field theory (CFT) as in [7], one can take $\epsilon = 0$. The k allows more generality for our main theorem and is natural in the setting of probability theory on spaces of temperate distributions, but it is not needed in usual QFT models. In order to work in the space $\mathcal{D}'(\mathbb{R}^d)$ of general Schwartz distributions, the previous bounds can be readily adapted by dropping k altogether and letting the constant K depend on the radius of a large ball in which all the points must be confined.

For further reference, we introduce the following terminology regarding the EFNNB. The points x_i or rather their *labels* i , $1 \leq i \leq m + n + p$, are called *effective* because they can be someone else's nearest neighbor and they all participate in the computation of the minimums over distance. By contrast, the points y_i or more precisely their labels i , $1 \leq i \leq m + n$, are called *virtual*. They only communicate with their x_i which serves as a local point of reference, the center of their “mini-universe”.

1.8. Worst-case scenario planning. For the bound on \mathcal{C}_{AB}^C , the first quantifier is “ \forall ” regarding the format, i.e., the choice of triple (A, B, C) . If one anticipates needing the bound for several but finitely many formats, it is advantageous to have the same K, k for all formats. Namely, the order of quantifiers can be modified to $\forall \epsilon > 0, \exists k \in \mathbb{N}, \exists K > 0, \forall$ format, etc. One just needs to take the largest k and K .

A similar property holds for the EFNNB. In this case a format consists in a choice of m, n, p , the A_i, B_i, Δ_i featuring in the OPE-like elements, and the choice of B_i 's for $m + 1 \leq i \leq m + n + p$. Let us denote such a format by F . First pick the smallest η and γ , namely, set $\eta = \min_F \eta_F$ and $\gamma = \min_F \gamma_F$. Then if one chooses $\epsilon > 0$, the indicator function on the left-hand side of (9) with the new η is bounded by the similar one for F . The correlator defined by the format F on the left-hand side is bounded as before and the old majorant with γ_F is converted into the new one with the uniform γ , at the cost of creating an additional factor

$$\prod_{i=1}^{m+n} \left\{ \frac{|y_i - x_i|}{\min_{j \neq i} |x_i - x_j|} \right\}^{\gamma_F - \gamma} \leq \eta^{(m+n)(\gamma_F - \gamma)}.$$

Finally, take $k = \max_F k_F$ and $K = \max_F [K_F \eta^{(m+n)(\gamma_F - \gamma)}]$ while keeping in mind that m, n depend on the format F . As a result, the bound (9) also works with the new order of quantifiers $\exists \eta > 0, \exists \gamma > 0, \forall \epsilon > 0, \exists k \in \mathbb{N}, \exists K > 0, \forall F$. By the same reasoning, one can restrict the common γ to be less than a specified positive number if needed, e.g., for local integrability reasons.

Remark 1. In §4 and §5 we will be rather implicit on how we do the worst-case scenario planning. We suggest to the reader to go over these sections a first time without worrying about how small γ and ϵ have to be and then come back to this section in order to see that the use of the same k and K is legitimate. The reason for this is that the number of formats involved in the proofs is finite, because \mathcal{A} is finite and the correlation in Proposition 1 has fixed order.

1.9. Non-degeneracy condition. In order to use the $\mathcal{O}_A \times \mathcal{O}_B$ OPE for the definition of a field \mathcal{O}_C , we need to be able to peel it off by a formula one may write intuitively as

$$\mathcal{O}_C \sim \frac{1}{\mathcal{C}_{AB}^C} \left[\mathcal{O}_A \times \mathcal{O}_B - \sum_{D \neq C} \mathcal{C}_{AB}^D \mathcal{O}_D \right]$$

and this requires a non-vanishing or non-degeneracy condition on the OPE coefficient \mathcal{C}_{AB}^C .

We say that the triple (A, B, C) is *non-degenerate* if $\forall \epsilon > 0, \exists k \in \mathbb{N}, \exists K > 0, \forall (x, y) \in \text{Conf}_2$,

$$\mathcal{C}_{AB}^C(x, y) \geq \frac{1}{K|x - y|^{[A]+[B]-[C]-\epsilon}} \times \langle x \rangle^{-k} \langle y \rangle^{-k} . \quad (11)$$

1.10. Statement of results. We assume that we have an OPE structure with alphabet \mathcal{A} and such that $[A] \in [0, \frac{d}{2})$ for all $A \in \mathcal{A}$. We assume the hypotheses from §1.6 and §1.7 hold for this OPE structure. We suppose that we have a subset $\mathcal{B} \subset \mathcal{A}$ together with a probabilistic incarnation $(\mathcal{O}_B)_{B \in \mathcal{B}}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for the subsystem of pointwise correlations corresponding to \mathcal{B} . We assume that C_* is an element of $\mathcal{A} \setminus \mathcal{B}$ such that $\mathcal{A}(\Delta) \setminus \{C_*\}$, where $\Delta = [C_*]$, is contained in \mathcal{B} . We also assume that we have two elements A, B in \mathcal{B} such that the triple (A, B, C_*) is non-degenerate. We let ρ be a mollifier as in §1.1, except that we now add the more restrictive hypotheses that ρ is compactly supported with $\text{supp } \rho \in \bar{B}(0, 1)$ (the closed Euclidean ball of radius one around the origin), and is pointwise nonnegative. We again use the notation $\rho_r(x) = L^{-rd} \rho(L^{-r}x)$ for the rescaled mollifier and use this to define the function

$$Z_r(x) = \left\{ \int_{\text{Conf}_2} dy \, dz \, \rho_r(x - y) \rho_r(x - z) \mathcal{C}_{AB}^{C_*}(y, z) \right\}^{-1}$$

in $\mathcal{O}_{M,x}(\mathbb{R}^d)$. We then introduce the random element of $\mathcal{O}_{M,x}(\mathbb{R}^d)$ given by

$$M_r(x) = Z_r(x) \left[\mathcal{O}_{A,r}(x) \mathcal{O}_{B,r}(x) - \sum_{C \in \mathcal{A}(\Delta) \setminus \{C_*\}} \tilde{\mathcal{O}}_{C,r}(x) \right] \quad (12)$$

where

$$\mathcal{O}_{A,r}(x) = (\mathcal{O}_A * \rho_r)(x) = \langle \mathcal{O}_A(y), \rho_r(x - y) \rangle_y$$

and similarly

$$\mathcal{O}_{B,r}(x) = (\mathcal{O}_B * \rho_r)(x) = \langle \mathcal{O}_B(z), \rho_r(x - z) \rangle_z$$

while

$$\tilde{\mathcal{O}}_{C,r}(x) = \langle \mathcal{O}_C(z), g_r(x, z) \rangle_z$$

with

$$g_r(x, z) = \rho_r(x - z) \times \int_{\mathbb{R}^d \setminus \{z\}} dy \, \rho_r(x - y) \mathcal{C}_{AB}^C(y, z) .$$

Note that the dependence on the sample $\omega \in \Omega$ has been suppressed from the notation and that $\mathcal{O}_A, \mathcal{O}_B, \mathcal{O}_C$ designate the distribution-valued random variables provided by the probabilistic incarnation for \mathcal{B} . We view $M_r(x)$ as the random Schwartz distribution whose action on a test function $f \in \mathcal{S}(\mathbb{R}^d)$ is of course given by

$$M_r(f) = \int_{\mathbb{R}^d} dx M_r(x) f(x) = \langle M_r(x), f(x) \rangle_x .$$

It is not trivial to show that $M_r(f)$ is indeed well defined, \mathcal{F} -measurable, and in every $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 1$. This will be done in §4. The main result in this article is as follows.

Theorem 1.

- (1) For any test function f , and when taking $r \rightarrow -\infty$, the random variable $M_r(f)$ converges in every $L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 1$, and \mathbb{P} -almost surely to a random variable which we will denote by $\mathcal{O}_{C_*}(f)$.
- (2) The limit is independent from the choice of mollifier ρ .
- (3) There exists a Borel-measurable map

$$\mathcal{P} : \prod_{C \in \mathcal{B}} \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$$

such that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{O}_{C_*}(f) = [\mathcal{P}((\mathcal{O}_C)_{C \in \mathcal{B}})](f)$$

\mathbb{P} -almost surely.

- (4) If one extends the probabilistic incarnation to $\mathcal{B} \cup \{C_*\}$ by adding the $\mathcal{S}'(\mathbb{R}^d)$ -valued random variable $\mathcal{P}((\mathcal{O}_C)_{C \in \mathcal{B}})$, then the result is a probabilistic incarnation of the system of pointwise correlations corresponding to the new set of labels $\mathcal{B} \cup \{C_*\}$.

Clearly, this can be iterated. By growing \mathcal{B} and also using the trivial construction of derivatives in the sense of Schwartz distributions of already existing random fields, one can construct all the composite operators of the fractional massless free field from §1.1 in this way, provided one remains under the Calderón-Zygmund $\frac{d}{2}$ threshold for scaling dimensions. This will be done in detail later in §7. Also, if one constructs say ϕ^4 as $\phi^2 \times \phi^2$ or as $\phi \times \phi^3$, the result is the same. Namely, our renormalized product construction is *associative*, as a trivial consequence of Theorem 1, Part (4).

1.11. Wider context and structure of the article.

1.11.1. *The OPE in mathematics and physics.* There exists several versions of the OPE. The one used here is the pointwise OPE in position or x -space. One can also express the OPE in a smeared sense, i.e., asymptotics such as (5) are to be interpreted in the sense of distributions involving suitable test functions [136, §2]. Finally, there is the OPE in Fourier or momentum space when modifying two momenta by adding ξ and $-\xi$ respectively and then taking the limit $|\xi| \rightarrow \infty$ (see, e.g., [141, §10.4]). The latter was historically important since it provides the theoretical counterpart (see, e.g., [27, Ch. 14]) of the SLAC experiments which confirmed the quark picture [21]. This experimental discovery has somewhat been eclipsed by the more recent ones regarding the Higgs boson [16, 26] and gravitational waves [1]. One should remember, however, that the deep inelastic scattering experiments were an important milestone in the progress of human knowledge, since they helped guide

researchers towards the discovery of asymptotic freedom and the elaboration of the Standard Model of particle physics. As to the pointwise OPE, it is a cornerstone of the conformal bootstrap (see, e.g., [138] for a pedagogical introduction in the 2D situation and [105, 117] for higher dimensions). Note that the pointwise OPE has led to important developments in mathematics too. The notions of vertex operator algebras (see, e.g., [45]), as well as chiral and factorization algebras [18] can be seen as ways of capturing the mathematical structure of the pointwise OPE. These mathematical applications pertain to algebraic geometry and representation theory or, more specifically, the area known as the geometric Langlands program (see, e.g., [44]). This article shows that the pointwise OPE is also important to *probability theory*. Our construction of the renormalized product \mathcal{O}_{C*} is the adaptation to the probability context of what is known as a “point-splitting procedure” in the QFT context [33, 59, 60, 125]. The theorem from §1.10 relies on a functional analytic part done in §4 using tools from what we call the Schwartz-Grothendieck-Fernique theory. It also relies on Proposition 1 which is a combinatorial estimate in the pure tradition of the École Polytechnique school of constructive QFT founded by Roland Sénéor. This estimate allows one to go from the pointwise OPE to a smeared OPE. It is therefore an Abelian theorem, from the point of view of Tauberian theory (see, e.g., [81]). Note that this theory has known a recent revival because of the needs of the OPE [127] (see also [126]). We remark that due to the availability of other methods for constructing products of random distributions [57, 56], it would be interesting to see if one could derive the pointwise OPE from the smeared one.

1.11.2. *Where our bounds and methods came from.* Let us now explain the origin of the BFNNB and EFNNB. They both were outcomes of the study of the p -adic or hierarchical fractional ϕ^4 model in [11]. *The result in [11], previously announced in [10], is the first rigorous mathematical substantiation of Wilson’s Nobel Prize winning ε Expansion [135] which gave “methods to calculate numerically the crucial quantities” [98], i.e., anomalous scaling dimensions.* In May 2015 (see [6, Theorem 3]), the author obtained the BFNNB with $\mathcal{A} = \{\mathbb{1}, \phi, \phi^2\}$ for the non-Gaussian hierarchical scaling limit constructed in [11], together with the local integrability proof in §2. The results from [11] give an explicit bi-infinite series over scales representation for the mixed ϕ and ϕ^2 correlations in terms of (operad-like) composition along a tree (as in [3, p. 203]) of certain maps. Among those, the most important one for the bounds is the one corresponding to degree two vertices, i.e., linear chains in the tree. This map, $\dot{V} \mapsto RG_{dv}[\vec{V}_*, \dot{V}]$ in the notation of [11, p. 126] is the renormalization group (RG) acting in the space \mathcal{E}_{pt} of “non-integrated” operator perturbations of the infrared fixed point \vec{V}_* . Namely, its differential at $\dot{V} = 0$, for a rescaling by a factor of L , would have the eigenvalue $L^{-[A]}$ for the operator \mathcal{O}_A instead of $L^{d-[A]}$. The (fusion) tree is determined by geometry, i.e., the relative positions of the points x_1, \dots, x_p for which the correlation is evaluated. The BFNNB simply comes from the estimates in [11] applied to the initial linear chains stemming from the leaves of the tree until the *first fusion* (vertex of the tree with degree ≥ 3) which corresponds to the scale given by the distance to the *nearest neighbor*. The bound comes in *factorized* form simply because the correlation is computed using a Fréchet differential of order p (the number of points) which is continuous, i.e., bounded by the product of norms of the inputs.

The contribution of the OPE-like factors in (9) comes from similar yet more involved RG arguments [8]. Showing that the EFNNB implies the analogue of Theorem 1 for $\mathcal{S}'(\mathbb{Q}_p^d)$ was done by the author in August 2015 [9]. The EFNNB over \mathbb{Q}_p looks exactly the same as the

one over \mathbb{R} given in (9), except for the *effect* of the CZ-like factors. The proof in the p -adic case *requires* CZ-like terms, i.e., “moving legs around” as in Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization in x -space [104, §II.2] or as in (17) below. However, the pointwise correlations outside the diagonal are smooth which in the p -adic setting means *locally constant*: the CZ-like elements have a vanishing contribution. In the real case, one can use the Fundamental Theorem of Calculus

$$\mathcal{O}_C(y) - \mathcal{O}_C(x) = \sum_{|\alpha|=1} \int_0^1 dt (y-x)^\alpha \partial^\alpha \mathcal{O}_C(x + t(y-x))$$

“inside correlations”. In the particular CFT case [7], where a derivative increases the scaling dimension by one, we see that such a CZ-element has the contribution written on the right-hand side of (9) with $\gamma = 1$. In order to make Theorem 1 more general, we also allowed $0 < \gamma < 1$. It turns out this kind of bound is exactly what is used in harmonic analysis and the theory of Calderón-Zygmund operators (see [32, p. 372] and [92, p. 9]), hence our choice of terminology. Note that *our* motivation for the present work is that it accomplishes *one* of the tasks from the program outlined in [7]: we showed how [7, Conjecture 8] implies [7, Conjecture 9].

1.11.3. *A conjecture and relation to work in perturbative QFT.* In [4, Conjecture 2.3], we gave a loosely formulated conjecture for historians of mathematics. Likewise, we propose the following conjecture for physicists.

Conjecture 1. (for physicists) *Any reasonable QFT satisfies the OPE in the strong sense expressed by the EFNNB (9), without the $\frac{d}{2}$ restriction on scaling dimensions.*

Note that the OPE is believed by physicists to be true (in the weaker sense of (5)) for any reasonable QFT. Because of this generality, one may wonder if the OPE would follow from the Wightman axioms (see, e.g., [121]), but this does not seem to be the case [107, 13]. See [108, 84, 86] for the better behaved case of CFT. Also note that the Wightman axioms are designed for flat space. Trying to find a suitable substitute that works for curved spaces has been a long-term pursuit, with decisive progress made in [68] (see [43] for a recent review). Reversing the question, Hollands and Wald proposed to solve this axiomatization problem in curved space by using the OPE itself as the axiom [69]. This led Hollands and his collaborators to develop a vigorous program for the mathematical study of the OPE from the perturbative QFT point of view, i.e., in the sense of formal power series [67, 63, 64, 65, 66, 47, 46]. This work uses the flow equation methods developed earlier by Keller and Kopfer [79, 80]. However, these are mainly Fourier or momentum space methods. For example, in [67], the authors prove the convergence of the OPE, at every finite order of perturbation theory, without limitation on the distance between the two points being collapsed. This is because the spectator fields are smeared with test functions that have compact support in Fourier space. Take the simple example from §1.1 and consider the partially smeared correlation $\langle \phi(x_1)\phi(x_2)\phi(\hat{f}_3)\phi(\hat{f}_4) \rangle$ where \hat{f}_3 and \hat{f}_4 are compactly supported. Then for fixed x_2 , this is real-analytic in x_1 on all of $\mathbb{R}^d \setminus \{x_2\}$. If \hat{f}_3 and \hat{f}_4 have constant modulus equal to one, as is the case when localizing at points x_3 and x_4 , then the singularity at $|x_1 - x_2| = \min\{|x_2 - x_3|, |x_2 - x_4|\}$ is indeed present. For the proof of the above conjecture, in perturbative QFT, which is formulated in x -space, we believe that position space methods are more appropriate. However, we have to admit that the above

mentioned works manage to go quite far! In particular, [47, Theorem 3] essentially proves the analogue of our conjecture above for the BFNNB. The bound [47, Theorem 3] is *not* in factorized form, but it is *grosso modo* equivalent to (10). We believe the most powerful techniques for proving our conjecture in the sense of perturbative QFT are the multiscale x -space methods for BPHZ renormalization developed by the École Polytechnique school of constructive QFT [39, 40]. The power of these methods resides in the fact they can be adapted (albeit *with tears*) to the non-perturbative situation [41, 87, 104, 12, 2, 89, 90, 124]. As a testimonial to this major contribution of Roland Sénéor, one could also mention that the methods from [39, 40] play an important role in the recent result by Chandra and Hairer announced in [24].

1.11.4. *Calculus IV and V.* What we called Calculus IV and V in [7] are respectively differential and integral calculus in infinite dimension. One could say that Calculus IV was initiated by Volterra [129, 130, 131], and Calculus V by Wiener [133]. They both fill the pages of QFT textbooks (see, e.g., [141, Ch. 5]). Multilinear algebra plays an important role in Calculus IV and V. In the finite dimensional situation, multilinear algebra can be seen as a game where one starts with some vector spaces X, Y, Z, \dots then one takes various tensor products and duals (denoted with a prime) such as $V = X \otimes Y' \otimes Z'$ and $W = Y \otimes Z \otimes X$. If one takes elements $A \in V$ and $B \in W$, then one can form a new one $A \bullet B \in X \otimes X$ using the duality pairings of Y with Y' and Z with Z' . Of course one can keep doing this forever. It is often important to not only know the final object obtained from this process as a tensor, or equivalently as some (multihomogeneous) polynomial function, but to also keep track of the entire assembly map that led to it. This usually involves a diagrammatic or graphical language with a modern category theoretic flavor as in, for instance, [116]. As explained in [4, §2], this game *together with its modern diagrammatic flavor* was born in the 1846 article [23] by Cayley. The main objects there are multihomogeneous forms which live in some tensor power of an n dimensional space X . They are contracted with Cayley's operator which lives in $\wedge^n X'$. The duality pairing is implemented via the duality between variables and differential operators. For example, the canonisant of the binary quintic represented by an explicit diagram in [4, Eq. 2.5] would be written as $\overline{12}^2 \overline{13}^2 \overline{23}^2$ in the notations of [23] (see [106] for more examples), and as $(ab)^2(ac)^2(bc)^2 a_x b_x c_x$ with the later German symbolic method. Without going into the precise definition of a compact closed category or a rigid symmetric monoidal category (see, e.g., [116]), one can say that such a multilinear algebraic game works so well for finite-dimensional spaces because several nice properties hold. For instance, one has the following facts.

(F1): One has a canonical isomorphism $(X')' \simeq X$.

(F2): One has a canonical isomorphism $\text{Hom}(X, Y) \simeq X' \otimes Y$.

(F3): If one takes an element $A \in X \otimes Y$ and an element $B \in Y' \otimes Z$, then the contraction $A \bullet B$ along the dual pair (Y, Y') can be seen, in a canonical way, as an element of $X \otimes Z$.

Also note that probability theory via the Chebyshev-Markov method of moments can be seen as an ultimate form of the game we mentioned. Indeed, the moments of a probability measure on X canonically live in the symmetric powers $\text{Sym}^n(X) \subset X^{\otimes n}$, involving all $n \in \mathbb{N}$. It is thus not surprising that diagrammatic methods are very useful in this context (see, e.g., [100]). Having mentioned all these trite facts about the finite-dimensional situation, one can now ask the following question.

Question: What are the *simplest* infinite-dimensional spaces for which one can play this multilinear algebra/probability theory game with *equal* success?

A first year graduate student would most likely reach for a simple space like the Hilbert space $X = L^2(\mathbb{R}^d) \simeq \ell^2(\mathbb{N})$, and try to build tensor products using the theory of second-quantization developed by Murray and von Neumann [97, Part I, Ch. II] and also Cook [29, 30]. This is the wrong answer. While (F1) and (F3) work well, (F2) fails catastrophically: the identity map in $\text{Hom}(X, X)$ does not belong to the proposed tensor product, since its kernel is given by a delta function. The failure of (F2) is very well explained in [50]: the tensor product used by Murray and von Neumann (and later Cook with a view to mathematical QFT) is not a true tensor product in the categorical sense.

The basic remark that the kernel of the identity is a Dirac delta function [111, p. 1] led Schwartz to the development of his monumental work [113, 114] on the *general* theory of distributional kernels to which Grothendieck contributed in a fundamental way [55]. We call this body of work the Schwartz-Grothendieck (SG) theory. It is customary to somewhat pompously call the particular case of (F2) with $X = \mathcal{D}(\mathbb{R}^m)$ and $Y = \mathcal{D}'(\mathbb{R}^n)$ “the Schwartz Kernel Theorem”. This is only the tip of the iceberg, and it is not terribly difficult. Indeed, simple proofs were given in [36, 51] and also in [118] for the case where $X = \mathcal{S}(\mathbb{R}^m)$ and $Y = \mathcal{S}'(\mathbb{R}^n)$. The most far-reaching part of SG theory resides in the functorial aspects of multilinear algebra with normal spaces of distributions (see, e.g., [71, p. 319] for a definition) or what Schwartz calls Volterra composition where the spaces X, Y , etc. can be of very different nature. As can be seen in §4, some of this deeper aspect of SG theory is needed in the context of this article. It is also needed for various applications: spectral theory [28], PDE theory [37], and systems engineering [139]. The canonical pedagogical reference for the theory of kernels is [123]. However, it is an introduction *to* rather than a treatment *of* general SG theory for which there is perhaps no source except the original works [55, 113, 114] (the recent book [15] comes close). Note that since $L^2(\mathbb{R}^d)$ is a normal space of distributions, the Murray-von Neumann-Cook approach to second quantization is a particular case of Volterra composition in SG theory.

We can now give the (two) correct answer(s) to the question above.

Answer: The spaces $\mathcal{S}(\mathbb{R}^d) \simeq \mathfrak{s}(\mathbb{N})$ and $\mathcal{S}(\mathbb{Q}_p^d) \simeq \mathfrak{s}_0(\mathbb{N})$, together with their topological duals.

Here we used

$$\mathfrak{s}(\mathbb{N}) = \{(x_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}} \mid \forall k, \exists K > 0, \forall n \geq 0, (n+1)^k |x_n| \leq K\}$$

to denote the space of rapidly decreasing sequences and

$$\mathfrak{s}_0(\mathbb{N}) = \{(x_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}} \mid \exists m \geq 0, \forall n \geq m, x_n = 0\}$$

to denote the space of almost finite sequences. Trying to break the tie between these two answers is perhaps a matter of psychology and prior training history. A number theorist might insist on not breaking the tie and considering $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{Q}_p^d)$ on perfectly equal footing. An analyst would perhaps prefer $\mathcal{S}(\mathbb{R}^d)$ because it is metrizable and more directly connected to applications in PDE theory, etc. An algebraist would perhaps prefer the non-metrizable space $\mathcal{S}(\mathbb{Q}_p^d)$ because it is an “ind-object” [78, §(0.4)]. Indeed, $\mathfrak{s}_0(\mathbb{N})$ is the simplest (strict) LF-space (see, e.g., [99, §3.8]). The author’s *opinion* on the matter should be quite clear from [5, 7]: the analyst is wrong and the algebraist is right. Namely, $\mathcal{S}(\mathbb{Q}_p^d)$ is simpler than $\mathcal{S}(\mathbb{R}^d)$. Our reason for this, however, is different from the algebraist’s. It is that $\mathcal{S}(\mathbb{Q}_p^d)$

provides, in a way that is principled and safe from modeling errors, a toy model for difficult analysis questions on $\mathcal{S}(\mathbb{R}^d)$, in the spirit of [122]. When analysts, e.g., use Walsh series or Mandelbrot cascades as toy models, they are secretly doing analysis in $\mathcal{S}(\mathbb{Q}_p^d)$ with $p = 2$.

As for what we call the Schwartz-Grothendieck-Fernique (SGF) theory, it is the combination of SG theory with probability theory on spaces of distributions. The latter was initiated by Itô [73] and Gel'fand [52]. A crucial contribution was added by Minlos [93], who proved the analogue of Bochner's Theorem, or essentially equivalently, showed how to go from the weak notion of generalized random field to the stronger notion of random distribution. However, the most accomplished presentation of this theory, is due to Fernique [42], who proved the analogue of the Levy Continuity Theorem. This explains our choice of terminology.

In §3 we will provide an elementary self-contained account of SGF theory in the particular case of temperate distributions. Unsurprisingly, what allows such simplification is the sequence space representation of $\mathcal{S}(\mathbb{R}^d)$ using Hermite functions [118]. This was of course known to Schwartz [110, p. 118] and Grothendieck [55, Ch. 2, p. 54], but they did not use it because they were aiming at much greater generality. In order to also deal with the more complicated space \mathcal{O}_M , we devised an approach which we cannot resist calling the “multiply and conquer” strategy: multiply by a generic function in \mathcal{S} , prove the needed kernel theorem or special case of [113, Proposition 34] only using \mathcal{S} and \mathcal{S}' , and finally undo the damage thanks to the multiplier space characterization of \mathcal{O}_M . Section 4 below gives an example of how this strategy works.

1.11.5. *Independence from the choice of mollifier.* The classic references [119, 54] on constructive QFT treat \mathcal{S}' as a measurable space without topology (the “ Q -space” coming from the diagonalization of a commutative von Neumann algebra). This does not take advantage of the probabilistic notions of tightness and weak convergence of probability measures, nor the relevant tools provided by Prokhorov's Theorem and the Levy Continuity Theorem, which have been worked out by Fernique in [42]. The main problem of constructive QFT is to study the weak convergence of certain probability measures on $\mathcal{S}'(\mathbb{R}^d)$. For concreteness, take the functional

$$V(\phi) = \alpha + \int_{\mathbb{R}^d} dx \{ \beta \phi(x)^2 + \gamma \phi(x)^4 \}$$

defined by the parameters $(\alpha, \beta, \gamma) \in \mathbb{R}^2 \times (0, \infty)$. This is a perfectly well-defined and continuous functional $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{R}$. However, trying to define the Gibbs, or Radon-Nikodym, perturbation $\exp(-V(\phi))d\mu(\phi)$ where μ is the law of the fractional massless free field from §1.1 typically leads to the problem addressed in this article, since the measure lives in \mathcal{S}' rather than \mathcal{S} . Therefore one has to proceed like physicists do, and follow the motto: *truncature et régularisation!* Let $\rho_{UV}, \rho_{IR} \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{\rho}_{UV}(0) = \rho_{IR}(0) = 1$, and, for $r, s \in \mathbb{Z}$, define the rescaled functions $\rho_{UV,r}(x) = L^{-rd}\rho_{UV}(L^{-r}x)$ and $\rho_{IR,s}(x) = \rho_{IR,s}(L^{-r}x)$. Finally, define the new field $\phi_{r,s} = \rho_{IR,s} \times (\phi * \rho_{UV,r})$ which is now in \mathcal{S} , and let $\mu_{r,s}$ denote the law of $\phi_{r,s}$ when ϕ is sampled according to μ . The main goal of the constructive field theorist is to analyze the weak convergence of $d\nu_{r,s}(\psi) = \exp(-V(\psi))d\mu_{r,s}(\psi)$ when $r \rightarrow -\infty$ and $s \rightarrow \infty$. This presentation is slightly different from the more usual one described in [7]. However, if for maximum safety one takes ρ_{UV} of compact support and $\rho_{IR} = 1$ in a neighborhood of the origin, then the difference is only at the boundary where $\rho_{IR,s}$ decays from 1 to 0. Of course α is determined by imposing that the 0-th moment has to be 1. Renormalization theory tells us we also have to allow β and γ to vary. In fact, they are also determined

essentially by fixing some well-chosen 2nd and 4th moments, i.e., renormalization conditions (see, e.g., [102, Eq. 2.56]), just as in the elementary central limit theorem in finite dimension (where it is the moments of order 0, 1, and 2 which are fixed).

The above is what we call the *Gibbsian setting* of traditional constructive QFT. In this setting life is very difficult because the probability measures $\nu_{r,s}$ are not in any obvious way the laws of $\mathcal{S}'(\mathbb{R}^d)$ -valued random variables on a *fixed probability space* $(\Omega, \mathcal{F}, \mathbb{R})$. Let us call the latter situation the *anchored setting*. If one is in the anchored situation, simple control on say 2nd moments or L^2 bounds implies convergence in probability and thus the weak convergence of the probability laws. This is probability textbook material in the case of \mathbb{R} -valued random variables, but it also works for $\mathcal{S}'(\mathbb{R}^d)$ -valued random variables as showed by Fernique [42]. The independence from the choice of ρ_{UV} in [57] or in Theorem 1, Part (2), is due to this anchored setting. When in the Gibbsian setting, this is very difficult to do. There is only one method available: showing Borel-summability of perturbation theory. This is another major contribution by Roland S  n  r [35, 88]. Also note that the stochastic quantization approach to constructive QFT is an important one because it realizes *the switch*: going from the Gibbsian setting to the anchored one. Indeed, the fixed probability space in that case is the space where the driving noise lives. This may seem as a mere trade-off since then one has to tackle the difficult problems of global-in-time analysis and the construction of the invariant measure. However, the results [95, 140, 58, 96] are very encouraging. Rigorous stochastic quantization was initiated by Jona-Lasinio and Mitter [74] (this was also the last research topic on which Roland S  n  r worked [75]). It has known a recent revival since the breakthrough by Hairer [57] which has been followed by two other treatments of the dynamic ϕ^4 model in three dimensions [22, 83].

1.11.6. *Structure of the article.* In §2 we show how the BFNNB implies local integrability for correlations. It uses elementary estimates and in particular Lemma 2 as well as Lemma 4 which are the workhorses on which the proof of Proposition 1 relies. These are very special cases of [57, Lemma 10.14] which is also a very busy workhorse when it comes to the applications of the theory of regularity structures (see, e.g., [72, §4.6.1]). The easy proof in §2 should give the gist of our method for proving Proposition 1 from which Theorem 1 follows. In §3 (to be included later), an elementary and self-contained presentation of SGF theory will be given in the *special case* of temperate distributions. In §4 based on SGF theory, we explain how to reduce the probability theory statements of Theorem 1 to the purely combinatorial estimate in Proposition 1. While SGF theory provides the functional analytic foundations for doing constructive QFT, we can say, paraphrasing [38], that this is not where the real action is. For the main result of this article, the action is in §5 where Proposition 1 is proved. In the rather short §6 we finish the proof of Theorem 1. Finally in §7 (to be included later), we will present a detailed special study of various CFTs in the sense of [7].

2. A WARM-UP EXAMPLE: LOCAL INTEGRABILITY

2.1. **Elementary beta integral estimates.** We show how the basic nearest neighbor bound implies the integrability property from §1.5. This uses some elementary lemmas. Throughout the remainder of this article, we will use the notation $\mathbb{1}\{\cdots\}$ for the sharp indicator function of the condition between braces.

Lemma 1. $\forall \alpha \in [0, d), \forall \beta \in (d, \infty), \exists K > 0, \forall x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d \setminus \{x\}} dy \langle y \rangle^{-\beta} |x - y|^{-\alpha} \leq K .$$

Proof: Since $\beta \geq d > 0$ and $\langle y \rangle \geq 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus \{x\}} dy \mathbb{1}\{|x - y| \leq 1\} \langle y \rangle^{-\beta} |x - y|^{-\alpha} \\ & \leq \int_{\mathbb{R}^d \setminus \{x\}} dy \mathbb{1}\{|x - y| \leq 1\} |x - y|^{-\alpha} = \frac{1}{d - \alpha} \times \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} . \end{aligned}$$

On the other hand, $\alpha \geq 0$ implies

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus \{x\}} dy \mathbb{1}\{|x - y| > 1\} \langle y \rangle^{-\beta} |x - y|^{-\alpha} \leq \\ & \int_{\mathbb{R}^d \setminus \{x\}} dy \mathbb{1}\{|x - y| \leq 1\} \langle y \rangle^{-\beta} \leq \int_{\mathbb{R}^d \setminus \{x\}} dy \langle y \rangle^{-\beta} . \end{aligned}$$

Combining both pieces gives the wanted bound with

$$K = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \left[\frac{1}{d - \alpha} + \int_0^\infty dr \frac{r^{d-1}}{(1 + r^2)^{\frac{\beta}{2}}} \right] .$$

□

The following lemma is the fundamental tool and resembles an estimate for the beta function.

Lemma 2. $\forall \alpha, \beta \in [0, \frac{d}{2}), \forall \gamma \in (d, \infty), \exists K > 0, \forall x, z \in \mathbb{R}^d,$

$$\int_{\mathbb{R}^d \setminus \{x, z\}} dy \langle y \rangle^{-\gamma} |x - y|^{-\alpha} |y - z|^{-\beta} \leq K$$

Note that we allow the case $x = z$ although it already follows from Lemma 1.

Proof: If $|x - y| \leq |y - z|$ then $|y - z|^{-\beta} \leq |x - y|^{-\beta}$ because $\beta \geq 0$. Therefore,

$$\begin{aligned} & \int_{\mathbb{R}^d \setminus \{x, z\}} dy \mathbb{1}\{|x - y| \leq |y - z|\} \langle y \rangle^{-\gamma} |x - y|^{-\alpha} |y - z|^{-\beta} \leq \\ & \int_{\mathbb{R}^d \setminus \{x, z\}} dy \mathbb{1}\{|x - y| \leq |y - z|\} \langle y \rangle^{-\gamma} |x - y|^{-(\alpha+\beta)} \leq \int_{\mathbb{R}^d \setminus \{x\}} dy \langle y \rangle^{-\gamma} |x - y|^{-(\alpha+\beta)} \leq K' \end{aligned}$$

where K' is provided by the previous lemma with $\alpha + \beta$ in lieu of α . By symmetry, the other piece

$$\int_{\mathbb{R}^d \setminus \{x, z\}} dy \mathbb{1}\{|x - y| > |y - z|\} \langle y \rangle^{-\gamma} |x - y|^{-\alpha} |y - z|^{-\beta}$$

is bounded by the same K' since $\alpha \geq 0$. The lemma follows with $K = 2K'$. □

We also need local versions of the two lemmas where the points are restricted to a closed Euclidean ball $\bar{B}(0, R)$ with $R > 0$. Note that the range of exponents is larger.

Lemma 3. $\forall \alpha \in (-\infty, d), \exists K > 0, \forall R > 0, \forall x \in \bar{B}(0, R),$

$$\int_{\bar{B}(0, R) \setminus \{x\}} dy |x - y|^{-\alpha} \leq KR^{d-\alpha}.$$

Proof: Since $|x - y| \leq |x| + |y| \leq 2R$, the integral is bounded by

$$\int_{\bar{B}(x, 2R) \setminus \{x\}} dy |x - y|^{-\alpha} = KR^{d-\alpha}$$

with

$$K = \frac{2^{d-\alpha}}{d-\alpha} \times \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

□

Lemma 4. $\forall \alpha, \beta \in (-\infty, \frac{d}{2}), \exists K > 0, \forall R > 0, \forall x, z \in \bar{B}(0, R),$

$$\int_{\bar{B}(0, R) \setminus \{x, z\}} dy |x - y|^{-\alpha} |y - z|^{-\beta} \leq KR^{d-\alpha-\beta}$$

Proof: Suppose first that $\alpha < 0$. Then, on the integration domain, $|x - y|^{-\alpha} \leq (2R)^{-\alpha}$ holds. Thus,

$$\int_{\bar{B}(0, R) \setminus \{x, z\}} dy |x - y|^{-\alpha} |y - z|^{-\beta} \leq (2R)^{-\alpha} \int_{\bar{B}(0, R) \setminus \{x, z\}} dy |y - z|^{-\beta}$$

and the previous lemma gives the desired bound. Likewise if $\beta < 0$ we use $|y - z|^{-\beta} \leq (2R)^{-\beta}$ and obtain the same conclusion. So we can assume that both α and β are nonnegative. We then have

$$\begin{aligned} & \int_{\bar{B}(0, R) \setminus \{x, z\}} dy \mathbb{1}\{|x - y| \leq |y - z|\} |x - y|^{-\alpha} |y - z|^{-\beta} \leq \\ & \int_{\bar{B}(0, R) \setminus \{x, z\}} dy \mathbb{1}\{|x - y| \leq |y - z|\} |x - y|^{-(\alpha+\beta)} \end{aligned}$$

and bound this with the previous lemma with $\alpha + \beta$ instead of α . The other piece

$$\begin{aligned} & \int_{\bar{B}(0, R) \setminus \{x, z\}} dy \mathbb{1}\{|x - y| > |y - z|\} |x - y|^{-\alpha} |y - z|^{-\beta} \leq \\ & \int_{\bar{B}(0, R) \setminus \{x, z\}} dy \mathbb{1}\{|x - y| \leq |y - z|\} |y - z|^{-(\alpha+\beta)} \end{aligned}$$

satisfies the same desired bound. □

2.2. The pin and sum argument with hairy cycles. We use the notation

$$||f||_{\alpha, k} = \sup_{x \in \mathbb{R}^d} \langle x \rangle^k |\partial^\alpha f(x)|$$

for the defining seminorms of $\mathcal{S}(\mathbb{R}^d)$ indexed by the integer $k \in \mathbb{N}$ and multiindex $\alpha \in \mathbb{N}^d$. Note that if $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ then $\langle x, y \rangle^2 = 1 + |x|^2 + |y|^2 \leq (1 + |x|^2)(1 + |y|^2)$. More generally, we have the following nice factorization property for concatenation of points or vectors

$$\langle x_1, \dots, x_n \rangle \leq \langle x_1 \rangle \cdots \langle x_n \rangle.$$

Let $f(x_1, \dots, x_p)$ be a function in $\mathcal{S}_{x_1, \dots, x_p}(\mathbb{R}^{pd})$. Then from the BFNNB with

$$\epsilon = \frac{1}{2} \min_{1 \leq i \leq p} \left(\frac{d}{2} - [B_i] \right)$$

we get

$$\int_{\text{Conf}_p} \prod_{i=1}^p dx_i |f(x_1, \dots, x_p)| \times |\langle \mathcal{O}_{B_1}(x_1) \cdots \mathcal{O}_{B_p}(x_p) \rangle| \leq K \|f\|_{0, k+d+1} \times \mathcal{I}$$

where

$$\mathcal{I} = \int_{\text{Conf}_p} \prod_{i=1}^p dx_i \prod_{i=1}^p \langle x_i \rangle^{-(d+1)} \prod_{i=1}^p \left(\min_{j \neq i} |x_i - x_j| \right)^{-([B_i] + \epsilon)}.$$

For $p = 1$, there is nothing to prove so we assume $p \geq 2$. Let \mathcal{N} denote the set of fixed-point-free endofunctions of $[p] = \{1, \dots, p\}$, namely, all maps $\tau : [p] \rightarrow [p]$ such that $\tau(i) \neq i$ for all i . Then for any fixed configuration of points $(x_1, \dots, x_p) \in \text{Conf}_p$, we have

$$1 \leq \sum_{\tau \in \mathcal{N}} \prod_{i=1}^p \mathbb{1} \left\{ |x_i - x_{\tau(i)}| = \min_{j \neq i} |x_i - x_j| \right\}.$$

Indeed, for each point i we can choose a nearest neighbor which we call $\tau(i)$. Insert the inequality inside the integral, then use the equalities in the indicator functions to replace the min's in the bound by $|x_i - x_{\tau(i)}|$'s. Then drop the indicator functions and pull the sum out of the integral. Therefore,

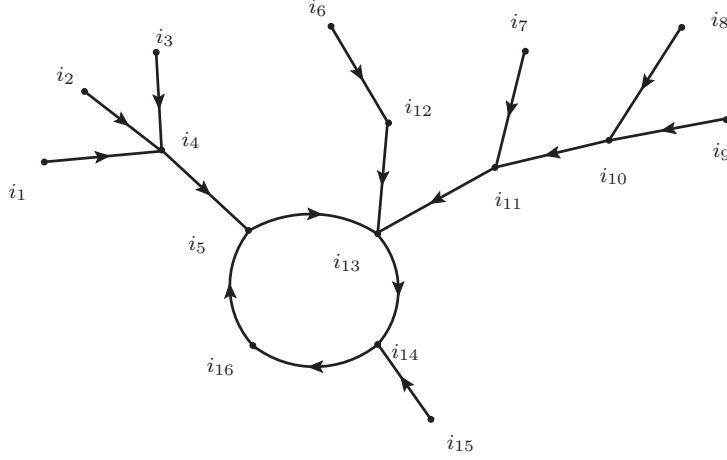
$$\mathcal{I} \leq \sum_{\tau \in \mathcal{N}} \int_{\text{Conf}_p} \prod_{i=1}^p dx_i \prod_{i=1}^p \langle x_i \rangle^{-(d+1)} \prod_{i=1}^p |x_i - x_{\tau(i)}|^{-([B_i] + \epsilon)}$$

One can draw a directed graph on the vertex set $[p]$ with edges $i \rightarrow \tau(i)$, for $i \in [p]$. Since τ is an endofunction, the connected components are what we call “hairy cycles”. Namely, each component is made of a central cycle playing the role of “root” for a collection of trees attached to it and oriented towards it. Then eliminate the trees by recursively using Lemma 1, starting from the leaves. The remaining cycle has length at least two by the fixed-point-free condition. Use Lemma 2 to open the cycle and erase two consecutive edges. Finally, eliminate the left over chain by Lemma 1 including its $\alpha = 0$ case for the last point. As a result

$$\int_{\text{Conf}_p} \prod_{i=1}^p dx_i |f(x_1, \dots, x_p)| \times |\langle \mathcal{O}_{B_1}(x_1) \cdots \mathcal{O}_{B_p}(x_p) \rangle| \leq O(1) \|f\|_{0, k+d+1}$$

and the function $\langle \mathcal{O}_{B_1}(x_1) \cdots \mathcal{O}_{B_p}(x_p) \rangle$ canonically defines an element of $\mathcal{S}'_{x_1, \dots, x_p}(\mathbb{R}^{pd})$. Note that we will use $O(1)$ in order to denote constants which do not need to be made explicit.

As a picture is worth a thousand words, let us show on an example how the pin and argument works.



Suppose for instance that the root is the label i_{16} . Namely, $x_{i_{16}}$ is the last variable to be integrated. Then a possible order of integration is given by the succession

$$i_1, i_2, i_3, i_4, i_6, i_{12}, i_7, i_8, i_9, i_{10}, i_{11}, i_{15}, i_5, i_{13}, i_{14}, i_{16}.$$

All these integrations are done with the help of Lemma 1, except when opening the cycle, i.e., integrating over x_{i_5} which uses Lemma 2.

Remark 2. *An important variant which will be used later is that one does not have to follow the arrows and pick the root in the cycle. One can be a contrarian and decide that the root is say i_{10} , i.e., that $x_{i_{10}}$ is integrated last. The argument works just as well, if one chooses for instance the order of integration*

$$i_9, i_8, i_7, i_6, i_{12}, i_{15}, i_1, i_2, i_3, i_4, i_{16}, i_5, i_{14}, i_{13}, i_{11}, i_{10}.$$

Indeed, when one is left with just the cycle attached by a path to the root, one can always open the cycle at a vertex which is different from the one where the path touches the cycle. This is again because the latter has length at least two.

Remark 3. *This kind of pin and sum argument is of course standard in constructive QFT [104]. However, the simplest example is perhaps the proof of the inverse function theorem for analytic functions given in [3, Theorem 1]. This is a baby finite-dimensional version of an idea which also works for PDEs as in [48, 120] for the Navier-Stokes Equation, or [25] for the nonlinear Schrödinger Equation.*

3. SCHWARTZ-GROTHENDIECK-FERNIQUE THEORY WITHOUT TEARS

3.1. General theory of kernels for \mathcal{S} and \mathcal{S}' .

3.2. Probability theory on \mathcal{S}' .

3.3. Variations on the multiplier space characterization of \mathcal{O}_M .

4. FROM PROBABILITY TO COMBINATORICS

We pick up the thread and notations from §1.10. Note that the function g_r from §1.10 can be written as

$$g_r(x, z) = \rho_r(x - z) \times \langle \mathcal{C}_{AB}^C(y, z), \rho_r(x - y) \rangle_y .$$

More generally, we can define

$$G_r(x, z, w) = \rho_r(x - z) \times \langle \mathcal{C}_{AB}^C(y, z), \rho_r(w - y) \rangle_y$$

for $(x, z, w) \in \mathbb{R}^{3d}$. Let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$, then

$$\psi_1(x)\psi_2(w)G_r(x, z, w) = \psi_1(x)\rho_r(x - z) \times \langle \mathcal{C}_{AB}^C(y, z), \psi_2(w)\rho_r(w - y) \rangle_y$$

belongs to $\mathcal{S}_{x,z,w}(\mathbb{R}^{3d})$, as results from our hypothesis (SH2) in §1.6. By restriction to the subspace $x = w$, we get that $\psi_1(x)\psi_2(x)g_r(x, z)$ belongs to $\mathcal{S}_{x,z}(\mathbb{R}^{2d})$. However, by a lemma of Miyazaki [94, Lemma 1] (see also [101, 128] or [49] which uses [132, Lemma 5]), every $\psi(x)$ in $\mathcal{S}_x(\mathbb{R}^d)$ can be written as $\psi_1(x)\psi_2(x)$. Thus, $\psi(x)g_r(x, z)$ belongs to $\mathcal{S}_{x,z}(\mathbb{R}^{2d})$ for all $\psi(x)$ in $\mathcal{S}_x(\mathbb{R}^d)$.

Now consider the integral used to define $Z_r(x)$, in §1.10, namely,

$$Y_r(x) = \int_{\text{Conf}_2} dy \, dz \, \rho_r(x - y)\rho_r(x - z) \, \mathcal{C}_{AB}^{C*}(y, z) = \langle 1, g_r(x, z) \rangle_z$$

in the special case $C = C_*$. If $\psi \in \mathcal{S}(\mathbb{R}^d)$, then

$$\psi(x)Y_r(x) = \langle 1, \psi(x)g_r(x, z) \rangle_z \in \mathcal{S}_x(\mathbb{R}^d) ,$$

because $\psi(x)g_r(x, z) \in \mathcal{S}_{x,z}(\mathbb{R}^{2d})$ and by Fubini's Theorem for distributions [109, Theorem IV, p. 108]. From the multiplier space characterization of \mathcal{O}_M (see [71, Proposition 5, p. 417] or [15, Proposition 1.6.1]), this implies that $Y_r(x)$ is in $\mathcal{O}_{M,x}(\mathbb{R}^d)$ and therefore smooth. For taking the inverse, the soft hypotheses from §1.6 are not enough. From (11) we get

$$Z_r(x)^{-1} \geq \int_{\text{Conf}_2} dy \, dz \, \rho_r(x - y)\rho_r(x - z) \frac{1}{K|y - z|^{[A]+[B]-[C_*]-\epsilon}} \langle y \rangle^{-k} \langle z \rangle^{-k}$$

or rather

$$Z_r(x)^{-1} \geq \int_{\text{Conf}_2} du \, dv \, \rho_r(u)\rho_r(v) \frac{1}{K|u - v|^{[A]+[B]-[C_*]-\epsilon}} \langle x - u \rangle^{-k} \langle x - v \rangle^{-k}$$

after the change of variables $y = x - u$ and $z = x - v$. Note that if $|u| \leq 1$ then

$$\langle x - u \rangle^2 \leq 1 + (|x| + 1)^2 \leq 1 + 2(|x|^2 + 1) \leq 3\langle x \rangle^2$$

so that $\langle x - u \rangle \leq \sqrt{3}\langle x \rangle$. From now on we assume $r \leq 0$. Because of the support condition on the mollifier ρ , this results in

$$Z_r(x)^{-1} \geq K^{-1}3^{-k}\langle x \rangle^{-2k} \int_{\text{Conf}_2} du \, dv \, \rho_r(u)\rho_r(v) \frac{1}{|u - v|^{[A]+[B]-[C_*]-\epsilon}} .$$

From the scaling change of variables $u = L^r \tilde{u}$ and $v = L^r \tilde{v}$ we obtain

$$Z_r(x)^{-1} \geq \tilde{K} \langle x \rangle^{-2k} L^{-r([A]+[B]-[C_*]-\epsilon)}$$

where

$$\tilde{K} = K^{-1}3^{-k} \int_{\text{Conf}_2} d\tilde{u} \, d\tilde{v} \, \rho(\tilde{u})\rho(\tilde{v}) |\tilde{u} - \tilde{v}|^{-[A]-[B]+[C_*]+\epsilon} > 0$$

from the assumptions on the mollifier ρ .

Thus

$$Z_r(x) \leq O(1)\langle x \rangle^{2k} L^{r([A]+[B]-[C_*]-\epsilon)}$$

Since derivatives of $Z_r(x) = Y_r(x)^{-1}$ are polynomials in $Z_r(x)$ and derivatives of $Y_r(r)$, we immediately conclude that $Z_r(x)$ belongs to $\mathcal{O}_{M,x}(\mathbb{R}^d)$.

For $f \in \mathcal{S}(\mathbb{R}^d)$ as in §1.10, let

$$\begin{aligned} M_{A,B,r}^C(f) &= \int_{\mathbb{R}^d} dx \ Z_r(x) \tilde{\mathcal{O}}_{C,r}(x) f(x) = \langle Z_r(x), \langle \mathcal{O}_C(z), g_r(x, z) \rangle_z f(x) \rangle_x \\ &= \langle Z_r(x), \langle \mathcal{O}_C(z), f(x) g_r(x, z) \rangle_z \rangle_x. \end{aligned}$$

From the above considerations, $f(x) g_r(x, z) \in \mathcal{S}_{x,z}(\mathbb{R}^{2d})$ and so by Fubini's Theorem for distributions $\langle \mathcal{O}_C(z), f(x) g_r(x, z) \rangle_z$ is in $\mathcal{S}_x(\mathbb{R}^d)$. Moreover,

$$M_{A,B,r}^C(f) = \langle Z_r(x) \otimes \mathcal{O}_C(z), f(x) g_r(x, z) \rangle_{x,z} = \langle \mathcal{O}_C(z), h_{A,B,r}^C(z) \rangle_z \quad (13)$$

where

$$h_{A,B,r}^C(z) = \langle Z_r(x), f(x) g_r(x, z) \rangle_x = \int_{\mathbb{R}^d} dx \ Z_r(x) f(x) g_r(x, z)$$

is a *fixed* test function in $\mathcal{S}_z(\mathbb{R}^d)$, while \mathcal{O}_C is random and depends on $\omega \in \Omega$. From our hypotheses on probabilistic incarnations and the representation (13), it follows that $M_{A,B,r}^C(f)$ is a well defined random variable on Ω with finite moments of all orders.

We now define

$$\begin{aligned} M_{A,B,r}(f) &= \int_{\mathbb{R}^d} dx \ Z_r(x) \mathcal{O}_{A,r}(x) \mathcal{O}_{B,r}(x) f(x) \\ &= \langle Z_r(x), \langle \mathcal{O}_A(y), \rho_r(x-y) \rangle_y \langle \mathcal{O}_B(z), \rho_r(x-z) \rangle_z f(x) \rangle_x \\ &= \langle Z_r(x) \otimes \mathcal{O}_A(y) \otimes \mathcal{O}_B(z), \rho_r(x-y) \rho_r(x-z) f(x) \rangle_{x,y,z} \end{aligned}$$

by Fubini's Theorem for distributions. This can be rewritten as

$$M_{A,B,r}(f) = \langle \mathcal{O}_A(y) \otimes \mathcal{O}_B(z), h_{A,B,r}(y, z) \rangle_{y,z} \quad (14)$$

where

$$h_{A,B,r}(y, z) = \int_{\mathbb{R}^r} dx \ Z_r(x) \rho_r(x-y) \rho_r(x-z) f(x)$$

is a fixed test function in $\mathcal{S}_{y,z}(\mathbb{R}^d)$. Since the tensor product of distributions is continuous for the strong topology, we have that $M_{A,B,r}(f)$ is \mathcal{F} -measurable. By [42, Theorem III.7.1] and its corollary, we also conclude that this random variable has moments of all orders.

In order to proceed further, we need to generalize the setting of §1.10 by juggling several renormalized product constructions at the same time.

For $1 \leq i \leq m$, we pick A_i, B_i in \mathcal{B} and $C_{*i} \in \mathcal{A} \setminus \mathcal{B}$. We set $\Delta_i = [C_{*i}]$ and assume as before that $\mathcal{A}(\Delta_i) \setminus \{C_{*i}\} \subset \mathcal{B}$. We pick C^∞ functions or mollifiers $\rho_{UV,i}$ or simply ρ_i such that $\text{supp } \rho_i \in \bar{B}(0, 1)$, $\rho_i \geq 0$, and $\int_{\mathbb{R}^d} dx \ \rho_i(x) = 1$. We pick shifts $\Delta r_i \in \{0, 1\}$. We also pick $n \geq m$ and $A_{m+1}, \dots, A_n \in \mathcal{A}$ for the spectator fields and we fix a collection of test functions f_1, \dots, f_n in $\mathcal{S}(\mathbb{R}^d)$. There will only be one varying quantity in the following discussion, namely the UV cut-off $r \in \mathbb{Z}$ which will be taken to $-\infty$. We define $r_i = r - \Delta r_i$ and the rescaled mollifiers $\rho_{i,r_i}(x) = L^{-dr_i} \rho_i(L^{-r_i} x)$. We set

$$Z_{i,r_i}(x) = \left\{ \int_{\text{Conf}_2} dy \ dz \ \rho_{i,r_i}(x-y) \rho_{i,r_i}(x-z) C_{A_i B_i}^{C_{*i}}(y, z) \right\}^{-1}$$

as well as $\mathcal{O}_{A_i, r_i}(x) = (\mathcal{O}_{A_i} * \rho_{i, r_i})(x) = \langle \mathcal{O}_{A_i}(y), \rho_{i, r_i}(x - y) \rangle_y$. We likewise set $\mathcal{O}_{B_i, r_i}(x) = (\mathcal{O}_{B_i} * \rho_{i, r_i})(x) = \langle \mathcal{O}_{B_i}(z), \rho_{i, r_i}(x - z) \rangle_z$. For $C \in \mathcal{A}(\Delta_i) \setminus \{C_{*i}\}$, we let

$$g_{i, r_i}(x, z) = \rho_{i, r_i}(x - z) \int_{\mathbb{R}^d \setminus \{z\}} dy \rho_{i, r_i}(x - y) \mathcal{C}_{A_i B_i}^C(y, z)$$

and define

$$\tilde{\mathcal{O}}_{C, r_i}(x) = \langle \mathcal{O}_C(z), g_{i, r_i}(x - z) \rangle_z$$

as before. This gives us the candidate for the regularized product “ $\mathcal{O}_{C_{*i}}$ ” as a function of x . Namely, it is

$$M_{i, r_i}(x) = Z_{i, r_i}(x) \left[\mathcal{O}_{A_i, r_i}(x) \mathcal{O}_{B_i, r_i}(x) - \sum_{C \in \mathcal{A} \setminus \{C_{*i}\}} \tilde{\mathcal{O}}_{C, r_i}(x) \right].$$

We let

$$M_{i, r_i}(f_i) = \langle M_{i, r_i}(x), f_i(x) \rangle_x = \int_{\mathbb{R}^d} dx M_{i, r_i}(x) f_i(x).$$

Our goal is to estimate the “moment” difference

$$\Upsilon_r = \text{TM}_r - \text{IPC}$$

where

$$\text{TM}_r = \mathbb{E} \left[\prod_{i=1}^m M_{i, r_i}(f_i) \times \prod_{i=m+1}^n \mathcal{O}_{A_i}(f_i) \right]$$

is a true moment, while

$$\text{IPC} = \int_{\text{Conf}_n} \prod_{i=1}^n dx_i \prod_{i=1}^n f_i(x_i) \times \left\langle \prod_{i=1}^m \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i=m+1}^n \mathcal{O}_{A_i}(x_i) \right\rangle$$

is an integral of pointwise correlations.

We have seen that $M_{i, r_i}(f_i)$ is a random variable in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ for all $p \geq 1$. By the multi-linear Hölder inequality, the random variable in the expectation defining TM_r is integrable. It is not hard to see that as a consequence of §2.2, the representations (13) and (14) for the pieces making up $M_{i, r_i}(f_i)$ and the discussion in [42, §II.2.5], that one has the following pointwise representation

$$\begin{aligned} \text{TM}_r &= \int_{\text{Conf}_{m+n}} \prod_{i=1}^m dy_i \prod_{i=1}^n dz_i \prod_{i=1}^m h_{i, r_i}(y_i, z_i) \times \prod_{i=m+1}^n f_i(z_i) \\ &\quad \times \left\langle \prod_{i=1}^m P_i(y_i, z_i) \prod_{i=m+1}^n \mathcal{O}_{A_i}(z_i) \right\rangle \end{aligned}$$

where, for $1 \leq i \leq m$, $P_i \in \mathcal{V}_2$ is given by

$$P_i = \mathcal{O}_{A_i} \otimes \mathcal{O}_{B_i} - \sum_{C_i \in \mathcal{A}(\Delta_i) \setminus \{C_{*i}\}} \mathcal{C}_{A_i B_i}^{C_i} \mathcal{O}_{\mathbb{1}} \otimes \mathcal{O}_{C_i}$$

and where

$$h_{i, r_i}(y_i, z_i) = \int_{\mathbb{R}^d} dx_i Z_{i, r_i}(x_i) f_i(x_i) \rho_{i, r_i}(x_i - y_i) \rho_{i, r_i}(x_i - z_i). \quad (15)$$

The proof of our main theorem from §1.10 is based on the follow proposition which is a purely combinatorial estimate.

Proposition 1. *The exists $\nu > 0$ such that*

$$|\Upsilon_r| \leq O(1) L^{\nu r}.$$

The proof of this proposition is provided in the next section. It can be viewed as an amplification of the one given in §2.

5. THE MAIN ESTIMATE

5.1. **Preparatory steps.** We define $Q_i \in \mathcal{V}_2$ by

$$Q_i = \mathcal{C}_{A_i B_i}^{C_{*i}} \mathcal{O}_1 \otimes \mathcal{O}_{C_{*i}}$$

so that

$$P_i = Q_i + R_{1,i} \tag{16}$$

where $R_{1,i}$ is the OPE-like element

$$R_{1,i} = \mathcal{O}_{A_i} \otimes \mathcal{O}_{B_i} - \sum_{C_i \in \mathcal{A}(\Delta_i)} \mathcal{C}_{A_i B_i}^{C_i} \mathcal{O}_1 \otimes \mathcal{O}_{C_i}.$$

By a decomposition (I_1, \dots, I_p) of a finite set I we mean an ordered collection of disjoint subsets whose union is I . This differs from a set partition because of the ordering and allowing the empty set. Borrowing our notation from the theory of symmetric functions, we will write $(I_1, \dots, I_p) \vdash I$ in order to say that (I_1, \dots, I_p) is a decomposition of I .

We now expand using (16) so that

$$\begin{aligned} \text{TM}_r = & \sum_{(I_1, I_{23}) \vdash [m]} \int_{\text{Conf}_{m+n}} \prod_{i=1}^m dy_i \prod_{i=1}^n dz_i \prod_{i=1}^m h_{i,r_i}(y_i, z_i) \times \prod_{i=m+1}^n f_i(z_i) \\ & \times \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_{23}} Q_i(y_i, z_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \end{aligned}$$

where $I_4 = [n] \setminus [m]$ is fixed.

We replace $h_{i,r_i}(y_i, z_i)$, for $i \in I_{23}$, by the integral in (15) which introduces $|I_{23}|$ new variables of integration x_i . We use the forgetful property (3) to replace $Q_i(y_i, z_i)$ by $\tilde{Q}_i(x_i, y_i, z_i)$ where $\tilde{Q}_i \in \mathcal{V}_3$ is given by $\mathcal{O}_1 \otimes Q_i$. We write $\hat{Q}_i = Q_i \otimes \mathcal{O}_1$ so that

$$\tilde{Q}_i = \hat{Q}_i + R_{2,i} \tag{17}$$

with

$$R_{2,i} = (1 \otimes \mathcal{C}_{A_i B_i}^{C_{*i}}) [\mathcal{O}_1 \otimes \mathcal{O}_1 \otimes \mathcal{O}_{C_{*i}} - \mathcal{O}_{C_{*i}} \otimes \mathcal{O}_1 \otimes \mathcal{O}_1]$$

where “1” simply is the function of the first argument x_i which is constant and equal to one.

We expand using (17) and get

$$\begin{aligned} \text{TM}_r = & \sum_{(I_1, I_2, I_3) \vdash [m]} \int_{\text{Conf}_{2|I_1|+3|I_2|+3|I_3|+|I_4|}} \prod_{i \in I_2 \cup I_3} dx_i \prod_{i \in [m]} dy_i \prod_{i \in [n]} dz_i \\ & \prod_{i \in I_1} h_{i,r_i}(y_i, z_i) \times \prod_{i \in I_4} f_i(z_i) \times \prod_{i \in I_2 \cup I_3} [Z_{i,r_i}(x_i) f_i(x_i) \rho_{i,r_i}(x_i - y_i) \rho_{i,r_i}(x_i - z_i)] \end{aligned}$$

$$\times \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} R_{2,i}(x_i, y_i, z_i) \times \prod_{i \in I_3} \widehat{Q}_i(x_i, y_i, z_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle$$

Since

$$\widehat{Q}_i(x_i, y_i, z_i) = \mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i) \mathcal{O}_{C_{*i}}(x_i)$$

“inside correlations”, we can factor $\mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i)$ out of the pointwise correlation and integrate over y_i, z_i for $i \in I_3$. This produces the inverse of $Z_{i,r_i}(x_i)$ by definition of the latter. Thus,

$$\begin{aligned} \text{TM}_r &= \sum_{(I_1, I_2, I_3) \vdash [m]} \int_{\text{Conf}_{2|I_1|+3|I_2|+|I_3|+|I_4|}} \prod_{i \in I_2 \cup I_3} dx_i \prod_{i \in I_1 \cup I_2} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\ &\quad \prod_{i \in I_1} h_{i,r_i}(y_i, z_i) \times \prod_{i \in I_4} f_i(z_i) \times \prod_{i \in I_3} f_i(x_i) \times \prod_{i \in I_2} [Z_{i,r_i}(x_i) f_i(x_i) \rho_{i,r_i}(x_i - y_i) \rho_{i,r_i}(x_i - z_i)] \\ &\quad \times \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} R_{2,i}(x_i, y_i, z_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \end{aligned}$$

We introduce the CZ-like elements $T_i = \mathcal{O}_{C_{*i}} \otimes \mathcal{O}_{\mathbf{1}} - \mathcal{O}_{\mathbf{1}} \otimes \mathcal{O}_{C_{*i}}$ for $i \in I_2$. Noting that one can write

$$R_{2,i}(x_i, y_i, z_i) = \mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i) [\mathcal{O}_{C_{*i}}(z_i) - \mathcal{O}_{C_{*i}}(x_i)] = \mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i) T_i(z_i, x_i)$$

“inside correlations”, we factor the $\mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i)$ ’s out of the pointwise correlation. We also insert the integrals (15) and therefore create $|I_1|$ new variables of integration x_i for $i \in I_1$. Therefore

$$\begin{aligned} \text{TM}_r &= \sum_{(I_1, I_2, I_3) \vdash [m]} \int_{\text{Conf}_{3|I_1|+3|I_2|+|I_3|+|I_4|}} \prod_{i \in I_1 \cup I_2 \cup I_3} dx_i \prod_{i \in I_1 \cup I_2} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\ &\quad \prod_{i \in I_2} \mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i) \times \prod_{i \in I_4} f_i(z_i) \times \prod_{i \in I_3} f_i(x_i) \times \prod_{i \in I_1 \cup I_2} [Z_{i,r_i}(x_i) f_i(x_i) \rho_{i,r_i}(x_i - y_i) \rho_{i,r_i}(x_i - z_i)] \\ &\quad \times \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} T_i(z_i, x_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \end{aligned}$$

Now note that IPC is the term corresponding to the decomposition $(I_1, I_2, I_3) = (\emptyset, \emptyset, [m])$. We now start putting absolute values inside the integral in order to write estimates. Hence

$$|\Upsilon_r| \leq \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \mathcal{P}^{(I)} \mathcal{I}^{(I)}$$

where $\mathcal{P}^{(I)} = 1$ and

$$\begin{aligned} \mathcal{I}^{(I)} &= \int_{\text{Conf}_{3|I_1|+3|I_2|+|I_3|+|I_4|}} \prod_{i \in I_1 \cup I_2 \cup I_3} dx_i \prod_{i \in I_1 \cup I_2} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\ &\quad \prod_{i \in I_2} |\mathcal{C}_{A_i B_i}^{C_{*i}}(y_i, z_i)| \times \prod_{i \in I_4} |f_i(z_i)| \times \prod_{i \in I_3} |f_i(x_i)| \times \prod_{i \in I_1 \cup I_2} |Z_{i,r_i}(x_i) f_i(x_i) \rho_{i,r_i}(x_i - y_i) \rho_{i,r_i}(x_i - z_i)| \\ &\quad \times \left| \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} T_i(z_i, x_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \right| \end{aligned}$$

and we suppressed the dependence on the decomposition from the notation for the prefactor $\mathcal{P}^{(I)}$ and the integral $\mathcal{I}^{(I)}$. There will be many more later, hence the roman numerals (I), (II), etc. We use the bound

$$|\rho_{i,r_i}(x_i - y_i)| \leq O(1)L^{-dr} \mathbb{1}\{|x_i - y_i| \leq L^r\}$$

and from now on we will denote r -independent constants by $O(1)$ if convenient. These constants can depend on everything else. For instance, the one above already ate up an L^d factor which is needed if $\Delta r_i = 1$.

In §4, we already established

$$Z_{i,r_i}(x) \leq O(1)\langle x \rangle^{2k} L^{r([A_i]+[B_i]-[C_{*i}]-\epsilon)}.$$

Inserting the previous bounds for the Z_{i,r_i} and the ρ_{i,r_i} as well as (8) for the $\mathcal{C}_{A_i B_i}^{C_{*i}}$ we get

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \mathcal{P}^{(II)} \mathcal{I}^{(II)}$$

with

$$\mathcal{P}^{(II)} = \prod_{i \in I_1 \cup I_2} L^{r([A_i]+[B_i]-\Delta_i-2d-\epsilon)}$$

and

$$\begin{aligned} \mathcal{I}^{(II)} = & \int_{\text{Conf}_{3|I_1|+3|I_2|+|I_3|+|I_4|}} \prod_{i \in I_1 \cup I_2 \cup I_3} dx_i \prod_{i \in I_1 \cup I_2} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\ & \prod_{i \in I_1 \cup I_2} \langle x_i \rangle^{2k} \times \prod_{i \in I_2} \langle y_i \rangle^k \times \prod_{i \in I_2} \langle z_i \rangle^k \times \prod_{i \in I_1 \cup I_2 \cup I_3} |f_i(x_i)| \times \prod_{i \in I_4} |f_i(z_i)| \\ & \times \prod_{i \in I_1 \cup I_2} (\mathbb{1}\{|x_i - y_i| \leq L^r\} \mathbb{1}\{|x_i - z_i| \leq L^r\}) \times \prod_{i \in I_2} |y_i - z_i|^{-[A_i]-[B_i]+\Delta_i-\epsilon} \\ & \times \left| \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} T_i(z_i, x_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \right| \end{aligned}$$

Now we make sure every point gets some long-distance decay by tapping into the nearest test function. Since $r \leq 0$ and $|u| \leq 1$ implies $\langle x - u \rangle \leq \sqrt{3}\langle x \rangle$ we can insert for all $i \in I_1$,

$$1 = \langle y_i \rangle^{-(d+1+k)} \langle y_i \rangle^{d+1+k} \leq O(1) \langle y_i \rangle^{-(d+1+k)} \langle x_i \rangle^{d+1+k}$$

and

$$1 = \langle z_i \rangle^{-(d+1+2k)} \langle z_i \rangle^{d+1+2k} \leq O(1) \langle z_i \rangle^{-(d+1+2k)} \langle x_i \rangle^{d+1+2k}$$

Likewise, for $i \in I_2$, we insert

$$1 = \langle y_i \rangle^{-k} \langle y_i \rangle^k \leq O(1) \langle y_i \rangle^{-k} \langle x_i \rangle^k$$

and

$$1 = \langle z_i \rangle^{-(d+1+2k)} \langle z_i \rangle^{d+1+2k} \leq O(1) \langle z_i \rangle^{-(d+1+2k)} \langle x_i \rangle^{d+1+2k}$$

Therefore

$$\begin{aligned} \mathcal{I}^{(II)} \leq & O(1) \int_{\text{Conf}_{3|I_1|+3|I_2|+|I_3|+|I_4|}} \prod_{i \in I_1 \cup I_2 \cup I_3} dx_i \prod_{i \in I_1 \cup I_2} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\ & \prod_{i \in I_1} \langle x_i \rangle^{2d+2+5k} \times \prod_{i \in I_2} \langle x_i \rangle^{d+1+5k} \times \prod_{i \in I_1} \langle y_i \rangle^{-(d+1+k)} \times \prod_{i \in I_1} \langle z_i \rangle^{-(d+1+2k)} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{i \in I_2} \langle z_i \rangle^{-(d+1+k)} \times \prod_{i \in I_1 \cup I_2 \cup I_3} |f_i(x_i)| \times \prod_{i \in I_4} |f_i(z_i)| \\
& \times \prod_{i \in I_1 \cup I_2} (\mathbb{1}\{|x_i - y_i| \leq L^r\} \mathbb{1}\{|x_i - z_i| \leq L^r\}) \times \prod_{i \in I_2} |y_i - z_i|^{-[A_i] - [B_i] + \Delta_i - \epsilon} \\
& \times \left| \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} T_i(z_i, x_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \right|
\end{aligned}$$

We now bound the test functions by suitable seminorms and prepare for the integration over points which are not in the big pointwise correlation, namely, x_i for $i \in I_1$ and y_i for $i \in I_2$.

For $i \in I_1$, we use the inequality

$$\begin{aligned}
& \langle x_i \rangle^{2d+2+5k} |f_i(x_i)| \mathbb{1}\{|x_i - y_i| \leq L^r\} \mathbb{1}\{|x_i - z_i| \leq L^r\} \leq \\
& \|f_i\|_{0,2d+2+5k} \mathbb{1}\{|x_i - z_i| \leq L^r\} \mathbb{1}\{|y_i - z_i| \leq 2L^r\}
\end{aligned}$$

For $i \in I_2$, we use the inequality

$$\begin{aligned}
& \langle x_i \rangle^{d+1+5k} |f_i(x_i)| \mathbb{1}\{|x_i - y_i| \leq L^r\} \mathbb{1}\{|x_i - z_i| \leq L^r\} \leq \\
& \langle x_i \rangle^{-(d+1+2k)} \|f_i\|_{0,2d+2+7k} \mathbb{1}\{|y_i - z_i| \leq 2L^r\} \mathbb{1}\{|x_i - z_i| \leq 2L^r\}
\end{aligned}$$

Note that we allow ourselves to loose a bit on the $|x_i - z_i|$ bound in order to make this case look like the previous one and thus ease the bookkeeping. For $i \in I_3$, we use the inequality

$$|f_i(x_i)| \leq \langle x_i \rangle^{-(d+1+2k)} \|f_i\|_{0,d+1+2k}$$

Finally, for $i \in I_4$, we use the inequality

$$|f_i(z_i)| \leq \langle z_i \rangle^{-(d+1+2k)} \|f_i\|_{0,d+1+2k}$$

Absorbing the Schwartz seminorms into the $O(1)$ constant, we obtain

$$\begin{aligned}
\mathcal{I}^{(\text{II})} & \leq O(1) \int_{\text{Conf}_{3|I_1|+3|I_2|+|I_3|+|I_4|}} \prod_{i \in I_1 \cup I_2 \cup I_3} dx_i \prod_{i \in I_1 \cup I_2} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\
& \prod_{i \in I_1} \langle y_i \rangle^{-(d+1+k)} \times \prod_{i \in I_1} \langle z_i \rangle^{-(d+1+2k)} \times \prod_{i \in I_2} \langle x_i \rangle^{-(d+1+2k)} \\
& \times \prod_{i \in I_2} \langle z_i \rangle^{-(d+1+k)} \times \prod_{i \in I_3} \langle x_i \rangle^{-(d+1+2k)} \times \prod_{i \in I_4} \langle z_i \rangle^{-(d+1+2k)} \\
& \times \prod_{i \in I_1} (\mathbb{1}\{|x_i - z_i| \leq L^r\} \mathbb{1}\{|y_i - z_i| \leq 2L^r\}) \times \prod_{i \in I_2} (\mathbb{1}\{|y_i - z_i| \leq 2L^r\} \mathbb{1}\{|x_i - z_i| \leq 2L^r\}) \\
& \times \prod_{i \in I_2} |y_i - z_i|^{-[A_i] - [B_i] + \Delta_i - \epsilon} \times \left| \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} T_i(z_i, x_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \right|
\end{aligned}$$

We now integrate over x_i , $i \in I_1$ and y_i , $i \in I_2$, respectively using the $\alpha = 0$ and $\alpha = [A_i] + [B_i] - \Delta_i + \epsilon$ cases of Lemma 3 Thus

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \mathcal{P}^{(\text{III})} \mathcal{I}^{(\text{III})}$$

with

$$\mathcal{P}^{(\text{III})} = \prod_{i \in I_1} L^{r([A_i] + [B_i] - \Delta_i - d - \epsilon)} \times \prod_{i \in I_2} L^{-r(d+2\epsilon)}$$

and

$$\begin{aligned} \mathcal{I}^{(\text{III})} = & \int_{\text{Conf}_{2|I_1|+2|I_2|+|I_3|+|I_4|}} \prod_{i \in I_2 \cup I_3} dx_i \prod_{i \in I_1} dy_i \prod_{i \in I_1 \cup I_2 \cup I_4} dz_i \\ & \prod_{i \in I_1} \langle y_i \rangle^{-(d+1+k)} \times \prod_{i \in I_1} \langle z_i \rangle^{-(d+1+2k)} \times \prod_{i \in I_2} \langle x_i \rangle^{-(d+1+2k)} \\ & \times \prod_{i \in I_2} \langle z_i \rangle^{-(d+1+k)} \times \prod_{i \in I_3} \langle x_i \rangle^{-(d+1+2k)} \times \prod_{i \in I_4} \langle z_i \rangle^{-(d+1+2k)} \\ & \times \prod_{i \in I_1} \mathbb{1}\{|y_i - z_i| \leq 2L^r\} \times \prod_{i \in I_2} \mathbb{1}\{|x_i - z_i| \leq 2L^r\} \\ & \times \left| \left\langle \prod_{i \in I_1} R_{1,i}(y_i, z_i) \times \prod_{i \in I_2} T_i(z_i, x_i) \times \prod_{i \in I_3} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_4} \mathcal{O}_{A_i}(z_i) \right\rangle \right| \end{aligned}$$

In order to continue, we will consolidate our notation by renaming the dummy variables of integration as follows

- For all $i \in I_1$, y_i stays y_i .
- For all $i \in I_1$, z_i becomes x_i .
- For all $i \in I_2$, z_i becomes y_i .
- For all $i \in I_2$, x_i stays x_i .
- For all $i \in I_3$, x_i stays x_i .
- For all $i \in I_4$, z_i becomes x_i .

We also introduce the notation $I_{34} = I_3 \cup I_4$ together with the new field labels D_i , $i \in I_{34}$ defined as follows.

- For all $i \in I_3$, $D_i = C_{*i}$.
- For all $i \in I_4$, $D_i = A_i$.

These changes made, we can give a simpler formula for the integral, i.e.,

$$\begin{aligned} \mathcal{I}^{(\text{III})} = & \int_{\text{Conf}_{2|I_1|+2|I_2|+|I_3|+|I_4|}} \prod_{i \in [n]} dx_i \prod_{i \in I_1 \cup I_2} dy_i \\ & \prod_{i \in [n]} \langle x_i \rangle^{-(d+1+2k)} \times \prod_{i \in I_1 \cup I_2} \langle y_i \rangle^{-(d+1+k)} \times \prod_{i \in I_1 \cup I_2} \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \\ & \times \left| \left\langle \prod_{i \in I_1} R_{1,i}(y_i, x_i) \times \prod_{i \in I_2} T_i(y_i, x_i) \times \prod_{i \in I_{34}} \mathcal{O}_{D_i}(x_i) \right\rangle \right| \end{aligned}$$

5.2. Re-expansion. At this point the pointwise correlation looks ready for the EFNNB. However, we are missing the indicator functions needed for this bound. We pick a number $\delta > 0$. For each i , $1 \leq i \leq n$, we insert in the integral $\mathcal{I}^{(\text{III})}$ the identity

$$1 = \mathbb{1} \left\{ \min_{j \neq i} |x_i - x_j| > \delta L^r \right\} + \mathbb{1} \left\{ \min_{j \neq i} |x_i - x_j| \leq \delta L^r \right\}$$

Then we expand. This results in a sum over decompositions (I_G, I_B) of $[n]$. Let $i \in [n]$. If we pick for it the first indicator function we say that i is good. If we pick the second, we say that i is bad. The set I_G is that of good labels, whereas I_B is the set of bad ones. The $R_{1,i}(y_i, x_i)$'s and $T_i(y_i, x_i)$ with $i \in I_G$ are left untouched. On the other hand, those for $i \in I_B$ must re-expanded completely. Recall that “inside correlations” we have

$$R_{1,i}(y_i, x_i) = \mathcal{O}_{A_i}(y_i) \mathcal{O}_{B_i}(x_i) - \sum_{C_i \in \mathcal{A}(\Delta_i)} \mathcal{C}_{A_i B_i}^{C_i}(y_i, x_i) \mathcal{O}_{C_i}(x_i) \quad (18)$$

and

$$T_i(y_i, x_i) = \mathcal{O}_{C_{*i}}(y_i) - \mathcal{O}_{C_{*i}}(x_i)$$

We need new notation for subsets of $[n]$, as follows.

- We let $I_{1G} = I_1 \cap I_G$.
- We let $I_{2G} = I_2 \cap I_G$.
- We let $I_{1\text{BOO}}$ be the set of i 's in $I_1 \cap I_B$ for which the first $\mathcal{O}\mathcal{O}$ term in (18) is chosen in the expansion.
- We let $I_{1\text{BCO}}$ be the set of i 's in $I_1 \cap I_B$ for which a $\mathcal{C}\mathcal{O}$ term in the sum in (18) is chosen for the expansion.
- We let $I_{2\text{BY}}$ be the set of i 's in $I_2 \cap I_B$ for which $\mathcal{O}_{C_{*i}}(y_i)$ is chosen.
- We let $I_{2\text{BX}}$ be the set of i 's in $I_2 \cap I_B$ for which $\mathcal{O}_{C_{*i}}(x_i)$ is chosen.

After putting absolute values so one does not have to worry about the signs produced by i 's in $I_{1\text{BCO}}$ or $I_{2\text{BX}}$, and after factoring the \mathcal{C} 's out of the pointwise correlation, the resulting estimate is

$$\begin{aligned} \mathcal{I}^{(\text{III})} &\leq \sum \int_{\text{Conf}_{2|I_1|+2|I_2|+|I_3|+|I_4|}} \prod_{i \in [n]} dx_i \prod_{i \in I_1 \cup I_2} dy_i \\ &\prod_{i \in [n]} \langle x_i \rangle^{-(d+1+2k)} \times \prod_{i \in I_1 \cup I_2} \langle y_i \rangle^{-(d+1+k)} \times \prod_{i \in I_1 \cup I_2} \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \\ &\times \prod_{i \in I_G} \mathbb{1} \left\{ \min_{j \neq i} |x_i - x_j| > \delta L^r \right\} \times \prod_{i \in I_B} \mathbb{1} \left\{ \min_{j \neq i} |x_i - x_j| \leq \delta L^r \right\} \\ &\times \prod_{i \in I_{1\text{BCO}}} |\mathcal{C}_{A_i B_i}^{C_i}(y_i, x_i)| \times \left| \left\langle \prod_{i \in I_{1G}} R_{1,i}(y_i, x_i) \times \prod_{i \in I_{2G}} T_i(y_i, x_i) \times \prod_{i \in I_{1\text{BOO}}} [\mathcal{O}_{A_i}(y_i) \mathcal{O}_{B_i}(x_i)] \right. \right. \\ &\quad \left. \left. \times \prod_{i \in I_{1\text{BCO}}} \mathcal{O}_{C_i}(x_i) \times \prod_{i \in I_{2\text{BY}}} \mathcal{O}_{C_{*i}}(y_i) \times \prod_{i \in I_{2\text{BX}}} \mathcal{O}_{C_{*i}}(x_i) \times \prod_{i \in I_{34}} \mathcal{O}_{D_i}(x_i) \right\rangle \right| \end{aligned}$$

For obvious reasons, we did not write the (formidable) summation index under the sum. Indeed, the sum is now over decompositions $(I_{1G}, I_{1\text{BOO}}, I_{1\text{BCO}})$ of I_1 , as well as decompositions $(I_{2G}, I_{2\text{BY}}, I_{2\text{BX}})$, of I_2 , followed by a summation over $C_i \in \mathcal{A}(\Delta_i)$, for each $i \in I_{1\text{BCO}}$.

At (long) last, we are now able to use the EFNNB as well as the (8) bound for the \mathcal{C} 's. Indeed, if we pick $\delta \geq 4\eta^{-1}$ then the indicator functions present imply the needed repulsive condition for the OPE-like elements $R_{1,i}(y_i, x_i)$, $i \in I_{1G}$, and CZ-like elements $T_i(y_i, x_i)$, $i \in I_{2G}$. Since the x_i 's and y_i 's are treated differently, and in order to keep the size of formulas under control, we need to introduce two notions of nearest-neighbor distance, before writing the outcome of the EFNNB.

For all $i \in [n] \setminus I_{2BY}$, we let

$$\text{NNDX}_i = \min \left\{ \min_{j \in ([n] \setminus I_{2BY}) \setminus \{i\}} |x_i - x_j|, \min_{j \in I_{1BOO} \cup I_{2BY}} |x_i - y_j| \right\}$$

For all $i \in I_{1BOO} \cup I_{2BY}$, we let

$$\text{NNDY}_i = \min \left\{ \min_{j \in [n] \setminus I_{2BY}} |y_i - x_j|, \min_{j \in (I_{1BOO} \cup I_{2BY}) \setminus \{i\}} |y_i - y_j| \right\}$$

These precautions taken, we now have

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \sum_{\substack{(I_{1G}, I_{1BOO}, I_{1BCO}) \vdash I_1 \\ (I_{2G}, I_{2BY}, I_{2BX}) \vdash I_2}} \sum_{(C_i)_{i \in I_{1BCO}} \in \prod_{i \in I_{1BCO}} \mathcal{A}(\Delta_i)} \mathcal{P}^{(IV)} \mathcal{I}^{(IV)}$$

with

$$\mathcal{P}^{(IV)} = \mathcal{P}^{(III)} = \prod_{i \in I_1} L^{r([A_i] + [B_i] - \Delta_i - d - \epsilon)} \times \prod_{i \in I_2} L^{-r(d + 2\epsilon)}$$

and

$$\begin{aligned} \mathcal{I}^{(IV)} &= \int_{\text{Conf}_{2|I_1|+2|I_2|+|I_3|+|I_4|}} \prod_{i \in [n]} dx_i \prod_{i \in I_1 \cup I_2} dy_i \\ &\prod_{i \in I_{1BCO}} \langle x_i \rangle^{-(d+1)} \times \prod_{i \in I_{2BY}} \langle x_i \rangle^{-(d+1+2k)} \times \prod_{i \in [n] \setminus (I_{1BCO} \cup I_{2BY})} \langle x_i \rangle^{-(d+1+k)} \\ &\times \prod_{i \in I_{2BX}} \langle y_i \rangle^{-(d+1+k)} \times \prod_{i \in (I_1 \cup I_2) \setminus I_{2BX}} \langle y_i \rangle^{-(d+1)} \\ &\times \prod_{i \in I_1 \cup I_2} \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \times \prod_{i \in I_G} \mathbb{1}\left\{ \min_{j \neq i} |x_i - x_j| > \delta L^r \right\} \times \prod_{i \in I_B} \mathbb{1}\left\{ \min_{j \neq i} |x_i - x_j| \leq \delta L^r \right\} \\ &\times \prod_{i \in I_{1BCO}} \frac{1}{|y_i - x_i|^{[A_i] + [B_i] - [C_i] + \epsilon}} \\ &\times \prod_{i \in I_{1G}} \frac{|y_i - x_i|^{\Delta_i + \gamma - [A_i] - [B_i]}}{\text{NNDX}_i^{\Delta_i + \gamma + \epsilon}} \times \prod_{i \in I_{2G}} \frac{|y_i - x_i|^\gamma}{\text{NNDX}_i^{\Delta_i + \gamma + \epsilon}} \times \prod_{i \in I_{1BOO}} \frac{1}{\text{NNDY}_i^{[A_i] + \epsilon} \times \text{NNDX}_i^{[B_i] + \epsilon}} \\ &\times \prod_{i \in I_{1BCO}} \frac{1}{\text{NNDX}_i^{[C_i] + \epsilon}} \times \prod_{i \in I_{2BY}} \frac{1}{\text{NNDY}_i^{\Delta_i + \epsilon}} \times \prod_{i \in I_{2BX}} \frac{1}{\text{NNDX}_i^{\Delta_i + \epsilon}} \times \prod_{i \in I_{34}} \frac{1}{\text{NNDX}_i^{[D_i] + \epsilon}} \end{aligned}$$

In order to reduce the complexity of this formula, we immediately integrate over the y_i 's with $i \in I_{1G} \cup I_{1BCO} \cup I_{2G} \cup I_{2BX}$. Indeed, the latter only couple to their corresponding x_i through a simple dependence that can be dealt with using Lemma 3.

For $i \in I_{1G}$, we use the bound

$$\int dy_i \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \langle y_i \rangle^{-(d+1)} |y_i - x_i|^{\Delta_i + \gamma - [A_i] - [B_i]} \leq O(1) L^{r(d + \Delta_i + \gamma - [A_i] - [B_i])}$$

which follows from $\langle y_i \rangle \geq 1$ and Lemma 3. For convenience, we did not write the domain of integration since it is \mathbb{R}^d minus a finite number of points (all the other x_j 's and y_j 's).

For $\in I_{1\text{BCO}}$, we similarly use the bound

$$\int dy_i \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \langle y_i \rangle^{-(d+1)} |y_i - x_i|^{-[A_i]-[B_i]+[C_i]-\epsilon} \leq O(1) L^{r(d-[A_i]-[B_i]+[C_i]-\epsilon)}.$$

For $\in I_{2\text{G}}$, we use the bound

$$\int dy_i \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \langle y_i \rangle^{-(d+1)} |y_i - x_i|^\gamma \leq O(1) L^{r(d+\gamma)}.$$

Finally, for $\in I_{2\text{BX}}$, we use the bound

$$\int dy_i \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \langle y_i \rangle^{-(d+1+k)} |y_i - x_i|^0 \leq O(1) L^{rd}.$$

As for the surviving long-distance decay factors, we bound them by the worst-case exponent, i.e., we turn them all into $\langle \cdot \rangle^{-(d+1)}$. Therefore

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \sum_{\substack{(I_{1\text{G}}, I_{1\text{BOO}}, I_{1\text{BCO}}) \vdash I_1 \\ (I_{2\text{G}}, I_{2\text{BY}}, I_{2\text{BX}}) \vdash I_2}} \sum_{(C_i)_{i \in I_{1\text{BCO}}} \in \prod_{i \in I_{1\text{BCO}}} \mathcal{A}(\Delta_i)} \mathcal{P}^{(\text{V})} \mathcal{I}^{(\text{V})}$$

with

$$\begin{aligned} \mathcal{P}^{(\text{V})} = & \prod_{i \in I_{1\text{G}}} L^{r(\gamma-\epsilon)} \times \prod_{i \in I_{1\text{BOO}}} L^{r([A_i]+[B_i]-\Delta_i-d-\epsilon)} \times \prod_{i \in I_{1\text{BCO}}} L^{r([C_i]-\Delta_i-2\epsilon)} \\ & \times \prod_{i \in I_{2\text{G}}} L^{r(\gamma-2\epsilon)} \times \prod_{i \in I_{2\text{BY}}} L^{-r(d+2\epsilon)} \times \prod_{i \in I_{2\text{BX}}} L^{-2\epsilon r} \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}^{(\text{V})} = & \int_{\text{Conf}_{n+|I_{1\text{BOO}}|+|I_{2\text{BY}}|}} \prod_{i \in [n]} dx_i \prod_{i \in I_{1\text{BOO}} \cup I_{2\text{BY}}} dy_i \\ & \prod_{i \in [n]} \langle x_i \rangle^{-(d+1)} \times \prod_{i \in I_{1\text{BOO}} \cup I_{2\text{BY}}} \langle y_i \rangle^{-(d+1)} \\ \times & \prod_{i \in I_{1\text{BOO}} \cup I_{2\text{BY}}} \mathbb{1}\{|y_i - x_i| \leq 2L^r\} \times \prod_{i \in I_{\text{G}}} \mathbb{1}\left\{\min_{j \neq i} |x_i - x_j| > \delta L^r\right\} \times \prod_{i \in I_{\text{B}}} \mathbb{1}\left\{\min_{j \neq i} |x_i - x_j| \leq \delta L^r\right\} \\ & \times \prod_{i \in I_{1\text{G}}} \frac{1}{\text{NNDX}_i^{\Delta_i+\gamma+\epsilon}} \times \prod_{i \in I_{2\text{G}}} \frac{1}{\text{NNDX}_i^{\Delta_i+\gamma+\epsilon}} \times \prod_{i \in I_{1\text{BOO}}} \frac{1}{\text{NNDY}_i^{[A_i]+\epsilon} \times \text{NNDX}_i^{[B_i]+\epsilon}} \\ & \times \prod_{i \in I_{1\text{BCO}}} \frac{1}{\text{NNDX}_i^{[C_i]+\epsilon}} \times \prod_{i \in I_{2\text{BY}}} \frac{1}{\text{NNDY}_i^{\Delta_i+\epsilon}} \times \prod_{i \in I_{2\text{BX}}} \frac{1}{\text{NNDX}_i^{\Delta_i+\epsilon}} \times \prod_{i \in I_{34}} \frac{1}{\text{NNDX}_i^{[D_i]+\epsilon}} \end{aligned}$$

Clearly, another round of consolidation of notation and relabeling is needed! We will avoid the different treatment of the x_i 's versus the y_i 's by replacing the set of labels $[n]$ by a larger one we will denote by V . The latter is a subset of $[n] \times \{\text{X}, \text{Y}\}$ where X, and Y are mere symbols. The set V is nothing more nor less than what we need in order to label all the variables in the previous integral. Thus, the set V is the (disjoint) union of the following ten blocks.

- Let $V_{1\text{G}}^{\text{X}} = I_{1\text{G}} \times \{\text{X}\}$.
- Let $V_{2\text{G}}^{\text{X}} = I_{2\text{G}} \times \{\text{X}\}$.

- Let $V_{1\text{BOO}}^Y = I_{1\text{BOO}} \times \{Y\}$
- Let $V_{1\text{BOO}}^X = I_{1\text{BOO}} \times \{X\}$
- Let $V_{1\text{BCO}}^X = I_{1\text{BCO}} \times \{X\}$.
- Let $V_{2\text{BY}}^Y = I_{2\text{BY}} \times \{Y\}$.
- Let $V_{2\text{BY}}^X = I_{2\text{BY}} \times \{X\}$.
- Let $V_{2\text{BX}}^X = I_{2\text{BX}} \times \{X\}$.
- Let $V_{34\text{G}}^X = (I_{34} \cap I_{\text{G}}) \times \{X\}$.
- Let $V_{34\text{B}}^X = (I_{34} \cap I_{\text{B}}) \times \{X\}$.

For each $a \in V$ we will define its corresponding (denominator) exponent β_a , we will also indicate its status, i.e., effective versus virtual from the point of view of the EFNNB we just used. This will result in a decomposition $(V_{\text{eff}}, V_{\text{virt}}) \vdash V$. Elements $a \in V$ which were not present in the correlation estimated by the EFNNB, namely, those of $V_{2\text{BY}}^X$, are declared virtual. The point is that only effective elements can be somebody else's neighbor for the calculation of the NNDX_i and NNDY_i functions. All of this is summarized in the following table which should help the reader follow the rest of the proof.

| Block | Status | $\beta_{(i,s)}$ |
|---------------------|--------|--------------------------------|
| $V_{1\text{G}}^X$ | eff | $\Delta_i + \gamma + \epsilon$ |
| $V_{2\text{G}}^X$ | eff | $\Delta_i + \gamma + \epsilon$ |
| $V_{1\text{BOO}}^Y$ | eff | $[A_i] + \epsilon$ |
| $V_{1\text{BOO}}^X$ | eff | $[B_i] + \epsilon$ |
| $V_{1\text{BCO}}^X$ | eff | $[C_i] + \epsilon$ |
| $V_{2\text{BY}}^Y$ | eff | $\Delta_i + \epsilon$ |
| $V_{2\text{BY}}^X$ | virt | 0 |
| $V_{2\text{BX}}^X$ | eff | $\Delta_i + \epsilon$ |
| $V_{34\text{G}}^X$ | eff | $[D_i] + \epsilon$ |
| $V_{34\text{B}}^X$ | eff | $[D_i] + \epsilon$ |

We let V^X denote the set of all $(i, s) \in V$ with $s = X$. We likewise define V^Y denote the set of all $(i, s) \in V$ with $s = Y$. We let V_{G} denote the set of all $(i, s) \in V$ with $i \in I_{\text{G}}$. We likewise define V_{B} as the set of all $(i, s) \in V$ with $i \in I_{\text{B}}$.

We also introduce notation for the following sets.

- We let $V_{\text{G}}^X = V_{\text{G}} \cap V^X = V_{1\text{G}}^X \cup V_{2\text{G}}^X \cup V_{34\text{G}}^X$.
- We let $V_{\text{B}}^X = V_{\text{B}} \cap V^X = V_{1\text{BOO}}^X \cup V_{2\text{BY}}^X \cup V_{2\text{BX}}^X \cup V_{34\text{B}}^X$.
- We let $V_{\text{B}}^Y = V_{\text{B}} \cap V^Y = V_{1\text{BOO}}^Y \cup V_{2\text{BY}}^Y$.

Of course, there is no need for a fourth set V_{G}^Y since it would be empty.

We define an involution $\iota : V \rightarrow V$ as follows.

- For all $i \in I_{1\text{BOO}} \cup I_{2\text{BY}}$, we let $\iota(i, Y) = (i, X)$ and $\iota(i, X) = (i, Y)$.
- For all $i \in [n] \setminus (I_{1\text{BOO}} \cup I_{2\text{BY}})$, we let $\iota(i, X) = (i, X)$

We can now rewrite the last integral as

$$\mathcal{I}^{(V)} = \int_{\text{Conf}_{|V|}} \prod_{a \in V} du_a$$

$$\prod_{a \in V} \langle u_a \rangle^{-(d+1)} \times \prod_{a \in V_{1\text{BOO}}^Y \cup V_{2\text{BY}}^Y} \mathbb{1}\{|u_a - u_{\iota(a)}| \leq 2L^r\}$$

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$$\begin{aligned}
& \times \prod_{a \in V_G^X} \mathbb{1} \left\{ \min_{b \in V^X \setminus \{a\}} |u_a - u_b| > \delta L^r \right\} \times \prod_{a \in V_B^X} \mathbb{1} \left\{ \min_{b \in V^X \setminus \{a\}} |u_a - u_b| \leq \delta L^r \right\} \\
& \times \prod_{a \in V_{\text{eff}}} \frac{1}{\left(\min_{b \in V_{\text{eff}} \setminus \{a\}} |u_a - u_b| \right)^{\beta_a}}
\end{aligned}$$

The next step which we call preemptive rerouting is necessary in order to keep the complexity of the graphs arising in the next section under control. We will overestimate the last integral by replacing V_{eff} simply by V . Indeed, $V \setminus V_{\text{eff}} = V_{2\text{BY}}^X$. There is no harm in adding the factors

$$\frac{1}{\left(\min_{b \in V \setminus \{a\}} |u_a - u_b| \right)^{\beta_a}}$$

for $a \in V_{2\text{BY}}^X$ since the corresponding exponent is $\beta_a = 0$. As for the elements $a \in V_{\text{eff}}$, we have

$$\frac{1}{\left(\min_{b \in V_{\text{eff}} \setminus \{a\}} |u_a - u_b| \right)^{\beta_a}} \leq \frac{1}{\left(\min_{b \in V \setminus \{a\}} |u_a - u_b| \right)^{\beta_a}}$$

since all exponents β_a in the table are nonnegative.

As a result,

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \sum_{\substack{(I_{1G}, I_{1\text{BOO}}, I_{1\text{BCO}}) \vdash I_1 \\ (I_{2G}, I_{2\text{BY}}, I_{2\text{BX}}) \vdash I_2}} \sum_{\substack{(C_i)_{i \in I_{1\text{BCO}}} \in \prod_{i \in I_{1\text{BCO}}} \mathcal{A}(\Delta_i)}} \mathcal{P}^{(\text{VI})} \mathcal{I}^{(\text{VI})}$$

with

$$\begin{aligned}
\mathcal{P}^{(\text{VI})} &= \mathcal{P}^{(\text{V})} = \prod_{i \in I_{1G}} L^{r(\gamma - \epsilon)} \times \prod_{i \in I_{1\text{BOO}}} L^{r([A_i] + [B_i] - \Delta_i - d - \epsilon)} \times \prod_{i \in I_{1\text{BCO}}} L^{r([C_i] - \Delta_i - 2\epsilon)} \\
&\times \prod_{i \in I_{2G}} L^{r(\gamma - 2\epsilon)} \times \prod_{i \in I_{2\text{BY}}} L^{-r(d + 2\epsilon)} \times \prod_{i \in I_{2\text{BX}}} L^{-2\epsilon r}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}^{(\text{VI})} &= \int_{\text{Conf}_{|V|}} \prod_{a \in V} du_a \\
&\prod_{a \in V} \langle u_a \rangle^{-(d+1)} \times \prod_{a \in V_{1\text{BOO}}^Y \cup V_{2\text{BY}}^Y} \mathbb{1}\{|u_a - u_{\iota(a)}| \leq 2L^r\} \\
&\times \prod_{a \in V_G^X} \mathbb{1} \left\{ \min_{b \in V^X \setminus \{a\}} |u_a - u_b| > \delta L^r \right\} \times \prod_{a \in V_B^X} \mathbb{1} \left\{ \min_{b \in V^X \setminus \{a\}} |u_a - u_b| \leq \delta L^r \right\} \\
&\times \prod_{a \in V} \frac{1}{\left(\min_{b \in V \setminus \{a\}} |u_a - u_b| \right)^{\beta_a}}
\end{aligned}$$

5.3. Graph construction. We are now ready for a more involved version of the argument in §2.2. We let \mathcal{G} be the set of all pairs (τ, σ) made of fixed-point-free endofunctions $\tau : V \rightarrow V$ and $\sigma : V_B \rightarrow V_B$ which satisfy the crucial requirement $\tau(V_B) \subset V_B$.

Claim: For any fixed point configuration $(u_a)_{a \in V} \in \text{Conf}_{|V|}$, the following inequality holds

$$\begin{aligned} 1 \leq & \sum_{(\tau, \sigma) \in \mathcal{G}} \prod_{a \in V} \mathbb{1} \left\{ |u_a - u_{\tau(a)}| = \min_{b \in V \setminus \{a\}} |u_a - u_b| \right\} \\ & \times \prod_{a \in V_B} \mathbb{1} \{ |u_a - u_{\sigma(a)}| \leq \delta L^r \} \\ & \times \prod_{a \in V_B} \mathbb{1} \{ |u_a - u_{\tau(a)}| \leq \delta L^r \} \end{aligned}$$

To prove the claim, we need to construct a pair (τ, σ) with the desired properties. For τ we proceed exactly as in §2.2, namely, we choose a nearest-neighbor. This already takes care of the first set of indicator functions. As for the construction of σ , it is done as follows. For $a \in V_B^Y$, we let $\sigma(a) = \iota(a) \in V_B^X$. For $a \in V_B^X$ we let $\sigma(a)$ be some choice of element $b \in V^X \setminus \{a\}$ such that $|u_a - u_b| \leq \delta L^r$. Since we can arrange for $\delta \geq 2$, this definition clearly satisfies all conditions in the second group of indicator functions. Note that such a b cannot belong to V_G^X by the repulsive condition in the definition of the latter. So σ is indeed an endofunction of V_B .

We are reduced to showing that, for any $a \in V_B$, we have $\tau(a) \in V_B$ and $|u_a - u_{\tau(a)}| \leq \delta L^r$. For $a \in V_B^Y$, $\sigma(a) \in V_B^X \subset V_B$ by construction, while the inequalities

$$2L^r \geq |u_a - u_{\iota(a)}| \geq \min_{b \in V \setminus \{a\}} |u_a - u_b| = |u_a - u_{\tau(a)}|$$

and $\delta \geq 2$ imply $|u_a - u_{\tau(a)}| \leq \delta L^r$.

Now let $a \in V_B^X$. Then $\sigma(a) \in V_B^X \setminus \{a\}$ satisfies

$$\delta L^r \geq |u_a - u_{\sigma(a)}| \geq \min_{b \in V \setminus \{a\}} |u_a - u_b| = |u_a - u_{\tau(a)}|$$

Finally, we argue by contradiction and assume that $\tau(a) \notin V_B$. Then $\tau(a) \in V_G^X$ which by definition implies

$$|u_{\tau(a)} - u_{\iota(a)}| > \delta L^r$$

because $\iota(a) \in V_B^X$ must be different from $\tau(a)$. On the other hand $\iota(a) \neq a$ satisfies

$$2L^r \geq |u_a - u_{\iota(a)}| \geq \min_{b \in V \setminus \{a\}} |u_a - u_b| = |u_a - u_{\tau(a)}|$$

while

$$|u_a - u_{\tau(a)}| \geq |u_{\tau(a)} - u_{\iota(a)}| - |u_a - u_{\iota(a)}| > (\delta - 2)L^r$$

which is a contradiction because we can arrange for $\delta \geq 4$.

Now that the claim is proved, we do as in §2.2. We pull the sum over (τ, σ) out of the integral, use the first set of indicator functions to replace the min's by $|u_a - u_{\tau(a)}|$'s. Finally, we discard the old indicator functions and the ones from the first group we just used, and we keep the new ones from the second and third group. As a result, we have

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \sum_{\substack{(I_{1G}, I_{1BOO}, I_{1BCO}) \vdash I_1 \\ (I_{2G}, I_{2BY}, I_{2BX}) \vdash I_2}} \sum_{(C_i)_{i \in I_{1BCO}} \in \prod_{i \in I_{1BCO}} \mathcal{A}(\Delta_i)}$$

$$\sum_{(\tau, \sigma) \in \mathcal{G}} \mathcal{P}^{(\text{VII})} \mathcal{I}^{(\text{VII})}$$

with

$$\begin{aligned} \mathcal{P}^{(\text{VII})} = \mathcal{P}^{(\text{VI})} &= \prod_{i \in I_{1G}} L^{r(\gamma - \epsilon)} \times \prod_{i \in I_{1BOO}} L^{r([A_i] + [B_i] - \Delta_i - d - \epsilon)} \times \prod_{i \in I_{1BCO}} L^{r([C_i] - \Delta_i - 2\epsilon)} \\ &\times \prod_{i \in I_{2G}} L^{r(\gamma - 2\epsilon)} \times \prod_{i \in I_{2BY}} L^{-r(d + 2\epsilon)} \times \prod_{i \in I_{2BX}} L^{-2\epsilon r} \end{aligned}$$

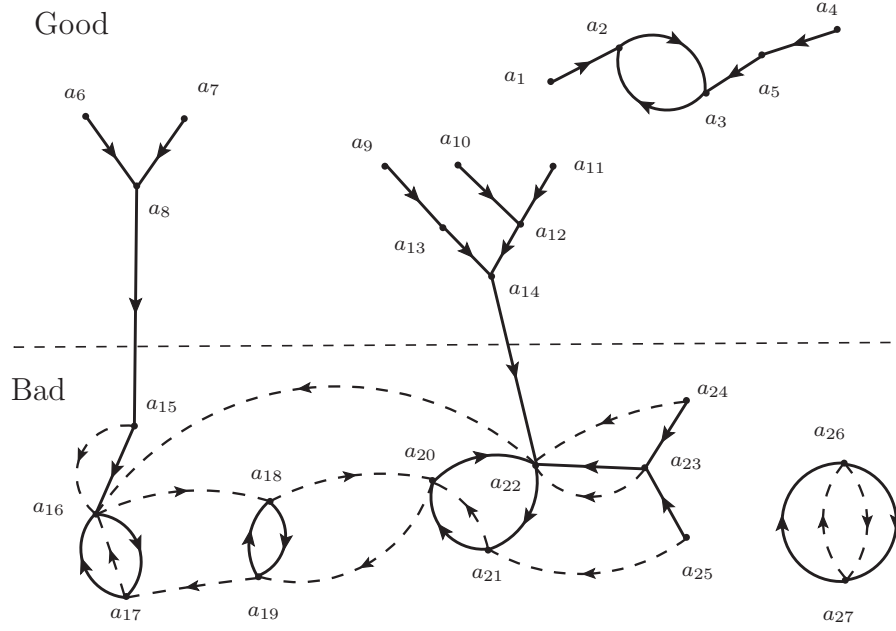
and

$$\begin{aligned} \mathcal{I}^{(\text{VII})} &= \int_{\text{Conf}_{|V|}} \prod_{a \in V} du_a \\ &\prod_{a \in V} \langle u_a \rangle^{-(d+1)} \times \prod_{a \in V_B} \mathbb{1}\{|u_a - u_{\sigma(a)}| \leq \delta L^r\} \\ &\times \prod_{a \in V_B} \mathbb{1}\{|u_a - u_{\tau(a)}| \leq \delta L^r\} \times \prod_{a \in V} |u_a - u_{\tau(a)}|^{-\beta_a} \end{aligned} \tag{19}$$

5.4. Two-scale pin and sum argument. The workhorses in §2.1 having now been provided with proper steering, we can sit back and watch them take care of the remaining integral. Indeed, one has a directed graph on V with two type of edges: $a \rightarrow \tau(a)$ and $a \rightarrow \sigma(a)$. Moreover, we have a decomposition $(V_G, V_B) \vdash V$. Clearly, the restriction of the graph to V_G is such that there can only be two kinds of connected components: 1) isolated pure τ hairy cycles or, 2) directed τ trees attached to some vertex in V_B . There is no σ edge incident to a vertex in V_G . Furthermore, the only possible communication between V_G and V_B is through an edge going from a vertex in V_G to a vertex in V_B , and this can only happen at most once per connected component of V_G . We therefore repeat the pin and sum argument from §2.2 and integrate over the vertices of V_G which produces an $O(1)$ factor.

Let $(W_1, \dots, W_q) \vdash V_B$ be a decomposition into connected components for the remaining graph made of both τ and σ edges. For i with $1 \leq i \leq q$, further decompose W_i into connected components for the subgraph only made of τ edges. This gives a decomposition $(W_{i,1}, \dots, W_{i,l_q}) \vdash W_i$. Then delete all σ edges internal to the $W_{i,j}$. Also, keep enough σ edges, i.e., $l_q - 1$ of them which connect $W_{i,1}, \dots, W_{i,l_q}$ together and delete the rest. When a σ edge is deleted we discard the corresponding indicator function in (19). Also pick some $b_i \in W_{i,1}$ which will serve as root for all of W_i . Discard the $\langle u_a \rangle^{-(d+1)}$ factors except those of b_1, \dots, b_q . As a result, each τ edge $a \rightarrow \tau(a)$ contributes an $L^{-\beta_{a\tau}}$ and each vertex contributes L^{dr} except the b_i 's which have to be integrated with Lemma 1 instead of Lemma 3.

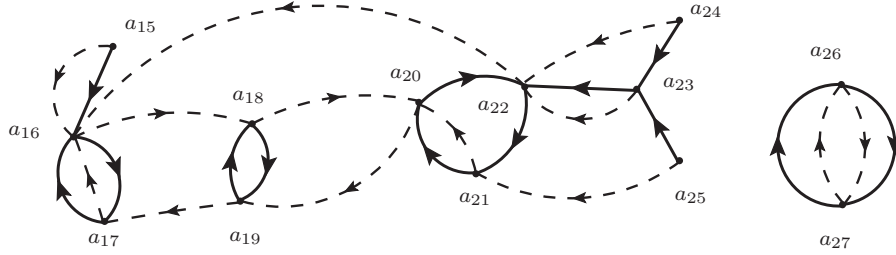
Again, an explicit example with a picture should help the reader follow the argument.



In the above graph, the solid lines represent τ edges and the dashed ones represent the σ edges. The elements in V_G are at the top and the ones in V_B are at the bottom. We first perform the succession of integrations corresponding to the sequence

$$a_1, a_4, a_5, a_3, a_2, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$$

in order to get rid of the good points. These operations all use Lemma 1, except the treatment of a_3 which uses Lemma 2. We are then left with the following picture.



In this example we have

$$W_1 = \{a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}\}$$

and

$$W_2 = \{a_{26}, a_{27}\} .$$

The sub-components are

$$W_{1,1} = \{a_{15}, a_{16}, a_{17}\} ,$$

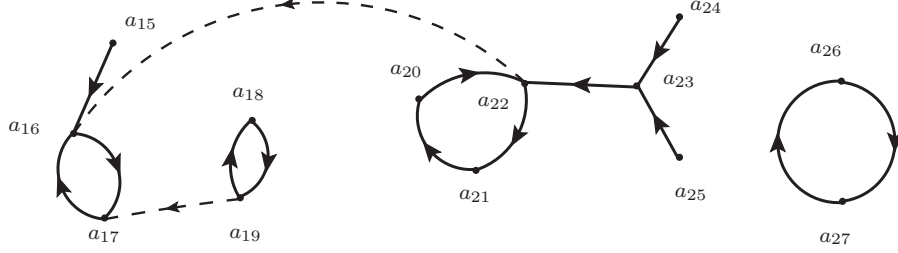
$$W_{1,2} = \{a_{18}, a_{19}\} ,$$

$$W_{1,3} = \{a_{20}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}\} ,$$

and

$$W_{2,1} = \{a_{26}, a_{27}\} .$$

Then we remove σ edges which do not improve connectivity, until we have a spanning tree between $W_{i,j}$'s in each W_i . Of course there are many ways to do so. For example, one could obtain the following graph.



For the choices of roots, we could for instance take $b_1 = a_{15}$ and $b_2 = a_{26}$. A possible order of integration is then

$$a_{27}, a_{26}, a_{24}, a_{25}, a_{23}, a_{20}, a_{21}, a_{22}, a_{18}, a_{19}, a_{17}, a_{16}, a_{15}.$$

Note that used the idea from Remark 2 in the last three integrations. For a_{15} and a_{26} , we used Lemma 1. For a_{27} , a_{20} , a_{18} and a_{17} , we used Lemma 4. Finally, for the remaining points, we used Lemma 3.

5.5. Power-counting. At this point we are left with an accounting problem which is to make sure the overall coefficient of r in the exponent of L is positive. Indeed,

$$|\Upsilon_r| \leq O(1) \sum_{\substack{(I_1, I_2, I_3) \vdash [m] \\ (I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])}} \sum_{\substack{(I_{1G}, I_{1BOO}, I_{1BCO}) \vdash I_1 \\ (I_{2G}, I_{2BY}, I_{2BX}) \vdash I_2}} \sum_{(C_i)_{i \in I_{1BCO}} \in \prod_{i \in I_{1BCO}} \mathcal{A}(\Delta_i)} \sum_{(\tau, \sigma) \in \mathcal{G}} \mathcal{P}^{(\text{VIII})} \mathcal{I}^{(\text{VIII})}$$

with $\mathcal{I}^{(\text{VIII})} = 1$ but also

$$\begin{aligned} \mathcal{P}^{(\text{VIII})} &= \prod_{i \in I_{1G}} L^{r(\gamma - \epsilon)} \times \prod_{i \in I_{1BOO}} L^{r([A_i] + [B_i] - \Delta_i - d - \epsilon)} \times \prod_{i \in I_{1BCO}} L^{r([C_i] - \Delta_i - 2\epsilon)} \\ &\times \prod_{i \in I_{2G}} L^{r(\gamma - 2\epsilon)} \times \prod_{i \in I_{2BY}} L^{-r(d + 2\epsilon)} \times \prod_{i \in I_{2BX}} L^{-2\epsilon r} \\ &\times L^{-rdq} \times \prod_{a \in V_B} L^{(d - \beta_a)r}. \end{aligned}$$

Using the table, this can be reorganized according to subsets of I as follows.

$$\mathcal{P}^{(\text{VIII})} = L^{r\alpha_{\text{Total}}}$$

with

$$\begin{aligned} \alpha_{\text{Total}} &= -dq + (\gamma - \epsilon)|I_{1G}| + (\gamma - 2\epsilon)|I_{2G}| \\ &+ \sum_{i \in I_{1BOO} \cup I_{1BCO} \cup I_{2BY} \cup I_{2BX}} (d - \Delta_i - 3\epsilon) + \sum_{i \in I_{34} \cap I_B} (d - [D_i] - \epsilon) \end{aligned}$$

Now notice that each connected component W_i must contain at least two elements of V_B^X . Indeed, it is not empty by definition. If it contains $a \in V_B^Y$ then by construction it also

contains $\sigma(a) \in V_B^X$. But if we caught one element b of V_B^X , then we can also catch another one, namely, $\sigma(b)$. Thus

$$2q \leq |V_B^X| = |I_{1BOO}| + |I_{1BCO}| + |I_{2BY}| + |I_{2BX}| + |I_{34} \cap I_B| = |I_B|$$

and therefore

$$\begin{aligned} \alpha_{\text{Total}} &\geq -\frac{d}{2}|I_B| + (\gamma - 2\epsilon)(|I_{1G}| + |I_{2G}|) \\ &\quad + |I_B| \times \min \left\{ \min_{i \in I_1 \cup I_2} (d - \Delta_i - 3\epsilon), \min_{i \in I_3} (d - \Delta_i - \epsilon), \min_{i \in I_4} (d - [A_i] - \epsilon) \right\} \\ &\geq (\gamma - 2\epsilon)(|I_{1G}| + |I_{2G}|) \\ &\quad + (|I_{1B}| + |I_{2B}| + |I_{34} \cap I_B|) \times \min \left\{ \min_{i \in I_1 \cup I_2 \cup I_3} \left(\frac{d}{2} - \Delta_i - 3\epsilon \right), \min_{i \in I_4} \left(\frac{d}{2} - [A_i] - \epsilon \right) \right\} \end{aligned}$$

We toss $|I_{34} \cap I_B|$ away and use $|I_1| = |I_{1G}| + |I_{1B}|$ as well as $|I_2| = |I_{2G}| + |I_{2B}|$ in order to write

$$\alpha_{\text{Total}} \geq \nu(|I_1| + |I_2|)$$

where

$$\nu = \min \left\{ \gamma - 2\epsilon, \min_{1 \leq i \leq m} \left(\frac{d}{2} - \Delta_i - 3\epsilon \right), \min_{m+1 \leq i \leq n} \left(\frac{d}{2} - [A_i] - \epsilon \right) \right\} > 0.$$

Recalling that $(I_1, I_2, I_3) \neq (\emptyset, \emptyset, [m])$ and $r \leq 0$, and all the sums being finite, we finally obtain

$$|\Upsilon_r| \leq O(1)L^{\nu r},$$

namely, the statement of Proposition 1. \square

6. PROOF OF THE MAIN THEOREM

We will be brief since now the rest of the proof of Theorem 1 is standard. For Part (1), it is enough to take $p \geq 2$ to be an even integer. We write

$$\begin{aligned} \|M_r(f) - M_{r-1}(f)\|_{L^p}^p &= \mathbb{E}[M_r(f) - M_{r-1}(f)]^p = \sum_{q=0}^p \binom{p}{q} (-1)^{p-q} \mathbb{E}[M_r(f)^{p-q} M_{r-1}(f)^q] \\ &= \sum_{q=0}^p \binom{p}{q} (-1)^{p-q} (\mathbb{E}[M_r(f)^{p-q} M_{r-1}(f)^q] - \text{IPC}) \end{aligned}$$

with

$$\text{IPC} = \int_{\text{Conf}_p} \prod_{i=1}^p dx_i \prod_{i=1}^p f(x_i) \left\langle \prod_{i=1}^p \mathcal{O}_{C_{*i}}(x_i) \right\rangle.$$

We apply Proposition 1 to each term and deduce convergence of the telescopic series in L^p , and thus in L^1 from which the almost sure convergence follows too. We then prove part (2). If one used two different mollifiers ρ_1 and ρ_2 and constructed two versions $\mathcal{O}_{C_{*,1}}(f)$ and $\mathcal{O}_{C_{*,2}}(f)$ of the smeared renormalized product corresponding to the label C_* , then

$$\|\mathcal{O}_{C_{*,1}}(f) - \mathcal{O}_{C_{*,2}}(f)\|_{L^2}^2 = \mathbb{E} \mathcal{O}_{C_{*,1}}(f)^2 - 2 \mathbb{E} \mathcal{O}_{C_{*,1}}(f) \mathcal{O}_{C_{*,2}}(f) + \mathbb{E} \mathcal{O}_{C_{*,2}}(f)^2 = 0$$

by a similar $r \rightarrow -\infty$ limit and application of Proposition 1.

It is also easy to similarly show that by adjoining the random variables $\mathcal{O}_{C_*}(f)$, $f \in \mathcal{S}(\mathbb{R}^d)$, the pointwise representation from §1.5 still holds. Hence $f \rightarrow \mathcal{O}_{C_*}(f)$ is continuous in say

L^2 and thus in probability. It is just a matter of collating this generalized random field into a random distribution (see [42, Proposition III.4.2 (a)]), in order to finish the proof of the theorem. \square

7. A SPECIAL STUDY OF VARIOUS CONFORMAL FIELD THEORIES

7.1. **An example: the fractional massless free field in any dimension.**

7.2. **A not-yet-example: the 3D fractional ϕ^4 model.**

7.3. **A non-example: the 2D Ising model.**

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ABDELMALEK ABDESSELAM, DEPARTMENT OF MATHEMATICS, P. O. BOX 400137, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904-4137, USA

E-mail address: malek@virginia.edu